AN EPIPERIMETRIC INEQUALITY APPROACH TO THE REGULARITY OF THE FREE BOUNDARY IN THE SIGNORINI PROBLEM WITH VARIABLE COEFFICIENTS

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ABSTRACT. In this paper we establish the $C^{1,\beta}$ regularity of the regular part of the free boundary in the Signorini problem for elliptic operators with variable Lipschitz coefficients. This work is a continuation of the recent paper [GSVG14], where two of us established the interior optimal regularity of the solution. Two of the central results of the present work are a new monotonicity formula and a new epiperimetric inequality.

CONTENTS

1. INTRODUCTION

1.1. **Statement of the problem and main assumptions.** The purpose of the present paper is to establish the $C^{1,\beta}$ regularity of the free boundary near so-called regular points in the Signorini problem for elliptic operators with variable Lipschitz coefficients. Although this work represents a continuation of the recent paper [GSVG14], where two of us established the interior optimal regularity of the solution, proving the regularity of the free boundary has posed some major new

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challenges. Two of the central results of the present work are a new monotonicity formula (Theorem 4.3) and a new epiperimetric inequality (Theorem 6.3). Both of these results have been inspired by those originally obtained by Weiss in [Wei99] for the classical obstacle problem, but the adaptation to the Signorini problem has required a substantial amount of new ideas.

The lower-dimensional (or *thin*) *obstacle problem* consists of minimizing the (generalized) Dirichlet energy

$$\min_{u\in\mathcal{K}}\int_\Omega \langle A(x)\nabla u,\nabla u\rangle dx, \tag{1.1}$$

where u ranges in the closed convex set

$$\mathcal{K}=\mathcal{K}_{g,\varphi}=\{u\in W^{1,2}(\Omega)\mid u=g \text{ on } \partial\Omega,\ u\geq\varphi \text{ on } \mathcal{M}\cap\Omega\}.$$

Here, $\Omega\subset\mathbb{R}^n$ is a given bounded open set, $\mathcal{M}$ is a codimension one manifold which separates Ω into two parts, g is a boundary datum and the function $\varphi:\mathcal{M}\to\mathbb{R}$ represents the lower-dimensional, or thin, obstacle. The functions g and φ are required to satisfy the standard compatibility condition $g\geq\varphi$ on $\partial\Omega\cap\mathcal{M}$. This problem is known also as (scalar) *Signorini problem*, as the minimizers satisfy Signorini conditions on $\mathcal{M}$ (see (1.7)–(1.9) below in the case of flat $\mathcal{M}$).

Our assumptions on the matrix-valued function $x\mapsto A(x)=[a_{ij}(x)]$ in (1.1) are that $A(x)$ is symmetric, uniformly elliptic, and Lipschitz continuous (in short $A\in C^{0,1}$). Namely:

$$a_{ij}(x)=a_{ji}(x)\quad \text{for } i,j=1,\dots,n, \text{ and every } x\in\Omega; \tag{1.2}$$

there exists $\lambda>0$ such that for every $x\in\Omega$ and $\xi\in\mathbb{R}^n$, one has

$$\lambda|\xi|^2\leq\langle A(x)\xi,\xi\rangle\leq\lambda^{-1}|\xi|^2; \tag{1.3}$$

there exists $Q\geq 0$ such that

$$|a_{ij}(x)-a_{ij}(y)|\leq Q|x-y|,\quad x,y\in\Omega. \tag{1.4}$$

By standard methods in the calculus of variations it is known that, under appropriate assumptions on the data, the minimization problem (1.1) admits a unique solution $u\in\mathcal{K}$, see e.g. [Fri88], or also [Tro87]. The set

$$\Lambda^\varphi(u)=\{x\in\mathcal{M}\cap\Omega\mid u(x)=\varphi(x)\}$$

is known as the *coincidence set*, and its boundary (in the relative topology of $\mathcal{M}$)

$$\Gamma^\varphi(u)=\partial_{\mathcal{M}}\Lambda^\varphi(u)$$

is known as the *free boundary.* In this paper we are interested in the local regularity properties of $\Gamma^\varphi(u)$. When $\varphi=0$, we will write $\Lambda(u)$ and $\Gamma(u)$, instead of $\Lambda^0(u)$ and $\Gamma^0(u)$.

We also note that since we work with Lipschitz coefficients, it is not restrictive to consider the situation in which the thin manifold is flat, which we take to be $\mathcal{M}=\{x_n=0\}$. We thus consider the Signorini problem (1.1) when the thin obstacle φ is defined on

$$B_1'=\mathcal{M}\cap B_1=\{(x',0)\in B_1\mid |x'|<1\},$$

which we call the thin ball in B_1. In this case we will also impose the following conditions on the coefficients

$$a_{in}(x',0)=0\quad \text{in } B_1', \text{ for } i<n, \tag{1.5}$$

which essentially means that the conormal directions $A(x',0)\nu_\pm$ are the same as normal directions $\nu_\pm=\mp e_n$. We stress here that the condition (1.5) is not restrictive as it can be satisfied by means of a $C^{1,1}$ transformation of variables, as proved in Appendix B of [GSVG14].

We assume that $\varphi \in C^{1,1}(B_1')$ and denote by u the unique solution to the minimization problem (1.1). (Notice that, by letting $\tilde{\varphi}(x', x_n) = \varphi(x')$, we can think of $\varphi \in C^{1,1}(B_1)$, although we will not make such distinction explicitly.) Such u satisfies

$$Lu = \operatorname{div}(A\nabla u) = 0 \quad \text{in } B_1^+ \cup B_1^-, \tag{1.6}$$

$$u - \varphi \geq 0 \quad \text{in } B_1', \tag{1.7}$$

$$\langle A\nabla u, \nu_+\rangle + \langle A\nabla u, \nu_-\rangle \geq 0 \quad \text{in } B_1', \tag{1.8}$$

$$(u - \varphi)(\langle A\nabla u, \nu_+\rangle + \langle A\nabla u, \nu_-\rangle) = 0 \quad \text{in } B_1'. \tag{1.9}$$

The conditions (1.7)–(1.9) are known as *Signorini* or *complementarity conditions*. It has been recently shown in [GSVG14] that, under the assumptions above, the unique solution of (1.6)–(1.9) is in $C^{1,1/2}(B_1^\pm \cup B_1')$. This regularity is optimal since the function $u(x) = \Re(x_1 + i|x_n|)^{3/2}$ solves the Signorini problem for the Laplacian and with thin obstacle $\varphi \equiv 0$.

Henceforth, we assume that 0 is a free boundary point, i.e., $0 \in \Gamma^\varphi(u)$, and we suppose without restriction that $A(0) = I$. Under these hypothesis, we consider the following *normalization* of u

$$v(x) = u(x) - \varphi(x') + bx_n, \quad b = \partial_{\nu_+} u(0). \tag{1.10}$$

Note that $\partial_{\nu_+} u(0) + \partial_{\nu_-} u(0) = 0$, which follows from (1.9), by taking a limit from inside the set $\{(x', 0) \in B_1' \mid u(x', 0) > \varphi(x')\}$, and from the hypothesis $A(0) = I$. This implies that

$$v(0) = |\nabla v(0)| = 0. \tag{1.11}$$

Next, note that, in view of (1.4) above and of the assumption $\varphi \in C^{1,1}(B_1)$, we have

$$-L(\varphi(x') - bx_n) \stackrel{\text{def}}{=} f \in L^\infty(B_1).$$

Hence, we can rewrite (1.6)–(1.9) in terms of v as follows:

$$Lv = \operatorname{div}(A\nabla v) = f \quad \text{in } B_1^+ \cup B_1^-, \quad f \in L^\infty(B_1), \tag{1.12}$$

$$v \geq 0 \quad \text{in } B_1', \tag{1.13}$$

$$\langle A\nabla v, \nu_+\rangle + \langle A\nabla v, \nu_-\rangle \geq 0 \quad \text{in } B_1', \tag{1.14}$$

$$v(\langle A\nabla v, \nu_+\rangle + \langle A\nabla v, \nu_-\rangle) = 0 \quad \text{in } B_1'. \tag{1.15}$$

Note that from (1.12) we have

$$\int_{B_1} (\langle A\nabla v, \nabla\eta\rangle + f\eta) = \int_{B_1'} (\langle A\nabla v, \nu_+\rangle + \langle A\nabla v, \nu_-\rangle)\eta, \quad \eta \in C_0^\infty(B_1). \tag{1.16}$$

and thus the conditions (1.13)–(1.15) imply that v satisfies the variational inequality

$$\int_{B_1} (\langle A\nabla v, \nabla(w - v)\rangle + f(w - v)) \geq 0, \quad \text{for any } w \in \mathcal{K}_{v,0},$$

where $\mathcal{K}_{v,0} = \{w \in W^{1,2}(B_1) \mid w = v \text{ on } \partial B_1, w \geq 0 \text{ on } B_1'\}$. Since $\mathcal{K}_{v,0}$ is convex, and so is the energy functional

$$\int_{B_1} (\langle A\nabla w, \nabla w\rangle + 2fw)\,, \tag{1.17}$$

this is equivalent to saying that v minimizes (1.17) among all functions $w \in \mathcal{K}_{v,0}$. Notice as well that

$$\Lambda^\varphi(u) = \{u(\cdot, 0) = \varphi\} = \{v(\cdot, 0) = 0\} = \Lambda^0(v) = \Lambda(v).$$

We will write $\Gamma^\varphi(u) = \Gamma^0(v) = \Gamma(v)$, and thus $0 \in \Gamma(v)$ now and (1.11) holds.

1.2. **Main result.** To state the main result of this paper, we need to further classify the free boundary points. This is achieved by means of the *truncated frequency function*

$$N(r) = N_L(v,r) \stackrel{\text{def}}{=} \frac{\sigma(r)}{2} e^{K' r^{\frac{1-\delta}{2}}} \frac{d}{dr} \log \max\left\{ \frac{1}{\sigma(r) r^{n-2}} \int_{S_r} v^2 \mu, r^{3+\delta} \right\}.$$

Here, $\mu(x) = \langle A(x)x, x\rangle/|x|^2$ is a conformal factor, $\sigma(r)$ is an auxiliary function with the property that $\sigma(r)/r \to \alpha > 0$ as $r \to 0$, $0 < \delta < 1$, and K' is a universal constant (see Section 2 for exact definitions and properties). This function was introduced in [GSVG14], and represents a version of Almgren's celebrated frequency function (see [Alm79]), adjusted for the solutions of (1.12)–(1.15). By Theorem 2.4 in [GSVG14], $N(r)$ is monotone increasing and hence the limit

$$\tilde{N}(0+) = \lim_{r\to 0} \tilde{N}(r), \quad \text{where} \quad \tilde{N}(r) = \frac{r}{\sigma(r)} N(r)$$

exists. The remarkable fact is that either $\tilde{N}(0+) = 3/2$, or $\tilde{N}(0+) \geq (3+\delta)/2$, see Lemma 2.5 below. This leads to the following definition.

Definition 1.1. We say that $0 \in \Gamma(v)$ is a *regular* point iff $\tilde{N}(0+) = 3/2$. Shifting the origin to $x_0 \in \Gamma(v)$, and denoting the corresponding frequency function by $\tilde{N}_{x_0}$, we define

$$\Gamma_{3/2}(v) = \{x_0 \in \Gamma(v) \mid \tilde{N}_{x_0}(0+) = 3/2\},$$

the set of all regular free boundary points, also known as the *regular set.*

The remaining part of the free boundary is divided into the sets $\Gamma_\kappa(v)$, according to the corresponding value of $\tilde{N}(0+) \stackrel{\text{def}}{=} \kappa$. We note that the range of possible values for κ can be further refined, provided more regularity is known for the coefficients $A(x)$. This can be achieved by replacing the truncation function $r^{3+\delta}$ in the formula for $N(r)$ with higher powers of r, similarly to what was done in [GP09] in the case of the Laplacian. This will provide more information on the set of possible values of $\tilde{N}(0+)$ which will serve as a classification parameter.

The following theorem is the central result of this paper.

Theorem 1.2. *Let v be a solution of* (1.12)–(1.15) *with $x_0 \in \Gamma_{3/2}(v)$. Then, there exists $\eta_0 > 0$, depending on x_0, such that, after a possible rotation of coordinate axes in $\mathbb{R}^{n-1}$, one has $B'_{\eta_0}(x_0) \cap \Gamma(v) \subset \Gamma_{3/2}(v)$, and*

$$B'_{\eta_0} \cap \Lambda(v) = B'_{\eta_0} \cap \{x_{n-1} \leq g(x_1, \dots, x_{n-2})\}$$

for $g \in C^{1,\beta}(\mathbb{R}^{n-2})$ with a universal exponent $\beta \in (0,1)$.

This result is known and well-understood in the case $L = \Delta$, see [ACS08] or Chapter 9 in [PSU12]. However, the existing proofs are based on differentiating the equation for v in tangential directions $e \in \mathbb{R}^{n-1}$ and establishing the nonnegativity of $\partial_e v$ in a cone of directions, near regular free boundary points (directional monotonicity). This implies the Lipschitz regularity of $\Gamma_{3/2}(v)$, which can be pushed to $C^{1,\beta}$ with the help of the boundary Harnack principle. The idea of the directional monotonicity goes back at least to the paper [Alt77], while the application of the boundary Harnack principle originated in [AC85]; see also [Caf98] and the book [PSU12]. In the case $L = \Delta$, we also want to mention two recent papers that prove the smoothness of the regular set: Koch, the second author, and Shi [KPS14] establish the real analyticity of $\Gamma_{3/2}$ by using hodograph-type transformation and subelliptic estimates, and De Silva and Savin [DSS14b] prove C^∞ regularity of $\Gamma_{3/2}$ by higher-order boundary Harnack principle in slit domains.

Taking directional derivatives, however, does not work well for the problem studied in this paper, particularly so since we are working with solutions of the non-homogeneous equation (1.12), which

corresponds to nonzero thin obstacle φ. In contrast, the methods in this paper are purely energy based, and they are new even in the case of the thin obstacle problem for the Laplacian. They are inspired by the homogeneity improvement approach of Weiss [Wei99] in the classical obstacle problem. The latter consists of a combination of a monotonicity formula and an epiperimetric inequality. In this connection we mention that recently in [FGS13], Focardi, Gelli, and Spadaro extended Weiss' method to the classical obstacle problem for operators with Lipschitz coefficients. We also mention a recent preprint by Koch, Rüland, and Shi [KRS15a] in which they use Carleman estimates to establish the almost optimal interior regularity of the solution in the variable coefficient Signorini problem, when the coefficient matrix is $W^{1,p}$, with $p > n+1$. In a personal communication [KRS15b] these authors have informed us of work in progress on the optimal interior regularity as well as the $C^{1,\beta}$ regularity of the regular set for $W^{1,p}$ coefficients with $p > 2(n+1)$. Their preprint was not available to us when the present paper was completed.

We next describe our proof of Theorem 1.2 above. The first main ingredient consists of the "almost monotonicity" of the Weiss-type functional

$$W_L(v,r) = \frac{1}{r^{n+1}} \int_{B_r} [\langle A(x)\nabla v, \nabla v\rangle + vf] - \frac{3/2}{r^{n+2}} \int_{S_r} v^2 \mu,$$

for solutions of (1.12)–(1.15). In Theorem 4.3 below we prove that $W_L(v,r) + Cr^{1/2}$ is nondecreasing for a universal constant C. Here, we were inspired by [Wei99] and [Wei98], where Weiss introduced related monotonicity formulas in the classical obstacle problem. In [GP09] two of us also proved a similar monotonicity formula in the Signorini setting, in the case of the Laplacian. In the present paper we use the machinery established in [GSVG14] to treat the case of variable Lipschitz coefficients. We mention that the geometric meaning of the functional W_L above is that it measures the closeness of the solution v to the prototypical homogeneous solutions of degree 3/2, i.e., the functions

$$a\Re(\langle x', \nu\rangle + i|x_n|)^{3/2}, \quad a \geq 0,\ \nu \in S_1.$$

The second central ingredient in the proof is the epiperimetric inequality for the functional

$$W(v) = W_\Delta(v,1) = \int_{B_1} |\nabla v|^2 - \frac{3}{2}\int_{S_1} v^2,$$

which states that if a (3/2)-homogeneous function w, nonnegative on B_1', is close to the solution $h(x) = \Re(x_1 + i|x_n|)^{3/2}$ in $W^{1,2}(B_1)$-norm, then there exists ζ in B_1 with $\zeta = w$ on ∂B_1 such that

$$W(\zeta) \leq (1-\kappa)W(w),$$

for a universal $0 < \kappa < 1$, see Theorem 6.3 below.

The combination of Theorems 4.3 and Theorem 6.3 provides us with a powerful tool for establishing the following geometric rate of decay for the Weiss functional:

$$W_L(v,r) \leq Cr^\gamma,$$

for a universal $\gamma > 0$. In turn, this ultimately implies that

$$\int_{S_1'} |v_{\bar{x},0} - v_{\bar{y},0}| \leq C|\bar{x} - \bar{y}|^\beta$$

for properly defined homogeneous blowups $v_{\bar{x},0}$ and $v_{\bar{y},0}$ at $\bar{x}, \bar{y} \in \Gamma_{3/2}(v)$. This finally implies the $C^{1,\beta}$ regularity of $\Gamma_{3/2}(v)$ in a more or less standard fashion.

1.3. **Structure of the paper.** The paper is organized as follows.

- In Section 2 we recall those definitions and results from [GSVG14] which constitute the background results of this paper.
- In Section 3 we give a more in depth look at regular free boundary points and prove some preliminary but important properties such as the relative openness of the regular set $\Gamma_{3/2}(v)$ in $\Gamma(v)$ and the local uniform convergence of the truncated frequency function $\tilde{N}_{\bar{x}}(r) \to 3/2$ on $\Gamma_{3/2}(v)$. We also introduce Almgren type scalings for the solutions, see (3.2), which play an important role in Section 7.
- In Section 4 we establish the first main technical tool of this paper, the Weiss-type monotonicity formula discussed above (Theorem 4.3). This result is instrumental to studying the homogeneous blowups of our function, which we do in Section 5.
- Section 6 is devoted to proving the second main technical result of this paper, the epiperimetric inequality (Theorem 6.3) which we have discussed above.
- Finally, in Section 7 we combine the monotonicity formula and the epiperimetric inequality to prove the main result of this paper, the $C^{1,\beta}$-regularity of the regular set $\Gamma_{3/2}(v)$ (Theorem 1.2).

2. Notation and preliminaries

2.1. **Basic notation.** Throughout the paper we use following notation. We work in the Euclidean space $\mathbb{R}^n$, $n \geq 2$. We write the points of $\mathbb{R}^n$ as $x = (x', x_n)$, where $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$. Very often, we identify the points $(x', 0)$ with x', thus identifying the "thin" space $\mathbb{R}^{n-1} \times \{0\}$ with $\mathbb{R}^{n-1}$.

For $x \in \mathbb{R}^n$, $x' \in \mathbb{R}^{n-1}$ and $r > 0$, we define the "solid" and "thin" balls

$$B_r(x) = \{y \in \mathbb{R}^n \mid |x - y| < r\}, \quad B'_r(x') = \{y' \in \mathbb{R}^{n-1} \mid |x' - y'| < r\}$$

as well as the corresponding spheres

$$S_r(x) = \{y \in \mathbb{R}^n \mid |x - y| = r\}, \quad S'_r(x') = \{y' \in \mathbb{R}^{n-1} \mid |x' - y'| = r\}.$$

We typically do not indicate the center, if it is the origin. Thus, $B_r = B_r(0)$, $S_r = S_r(0)$, etc. We also denote

$$B_r^{\pm}(x', 0) = B_r(x', 0) \cap \{\pm x_n > 0\}, \quad \mathbb{R}^n_{\pm} = \mathbb{R}^n \cap \{\pm x_n > 0\}.$$

For a given direction e, we denote the corresponding directional derivative by

$$\partial_e u = \langle \nabla u, e \rangle,$$

whenever it makes sense. For the standard coordinate directions $e = e_i$, $i = 1, \ldots, n$, we also abbreviate $\partial_i u = \partial_{e_i} u$.

In the situation when a domain $\Omega \subset \mathbb{R}^n$ is divided by a manifold $\mathcal{M}$ into two subdomains Ω_+ and Ω_-, ν_+ and ν_- stand for the exterior unit normal for Ω_+ and Ω_- on $\mathcal{M}$. Moreover, we always understand $\partial_{\nu_+} u$ ($\partial_{\nu_-} u$) on $\mathcal{M}$ as the limit from within Ω_+ (Ω_-). Thus, when $\Omega_\pm = B_r^\pm$, we have

$$\nu_\pm = \mp e_n, \quad -\partial_{\nu_+} u(x', 0) = \lim_{\substack{y \to (x', 0) \\ y \in B_r^+}} \partial_n u \stackrel{\text{def}}{=} \partial_n^+ u(x', 0), \quad \partial_{\nu_-} u(x', 0) = \lim_{\substack{y \to (x', 0) \\ y \in B_r^-}} \partial_n u \stackrel{\text{def}}{=} \partial_n^- u(x', 0)$$

In integrals, we often do not indicate the measure of integration if it is the Lebesgue measure on subdomains of $\mathbb{R}^n$, or the Hausdorff $\mathcal{H}^k$ measure on manifolds of dimension k.

Hereafter, when we say that a constant is universal, we mean that it depends exclusively on n, on the ellipticity bound λ on $A(x)$, and on the Lipschitz bound Q on the coefficients $a_{ij}(x)$. Likewise, we will say that $O(1)$, $O(r)$, etc, are universal if $|O(1)| \leq C$, $|O(r)| \leq Cr$, etc, with $C \geq 0$ universal.

2.2. **Summary of known results.** For the convenience of the reader, in this section we briefly recall the definitions and results proved in [GSVG14] which will be used in this paper.

As stated in Section 1, we work under the nonrestrictive situation in which the thin manifold $\mathcal{M}$ is flat. More specifically, we consider the Signorini problem (1.1) when the thin obstacle φ is defined on

$$B_1' = \mathcal{M} \cap B_1 = \{(x', 0) \in B_1 \mid |x'| < 1\}.$$

We assume that $\varphi \in C^{1,1}(B_1')$ and denote by u the unique solution to the minimization problem (1.1), which then satisfies (1.6)–(1.9). We assume that 0 is a free boundary point, i.e., $0 \in \Gamma^{\varphi}(u)$, and that $A(0) = I$, and we consider the normalization of u as in (1.10), i.e.,

$$v(x) = u(x) - \varphi(x') + bx_n, \quad \text{with } b = \partial_{\nu_+} u(0).$$

As remarked on Section 1, v satisfies (1.12)–(1.15) with $f \overset{\text{def}}{=} -L(\varphi(x') - bx_n) \in L^\infty(B_1)$, and has the additional property that $v(0) = |\nabla v(0)| = 0$, see (1.11) above.

We then recall the following definitions from [GSVG14]. The *Dirichlet integral* of v in B_r is defined by

$$D(r) = D_L(v, r) = \int_{B_r} \langle A(x) \nabla v, \nabla v \rangle,$$

and the *height function* of v in S_r is given by

$$H(r) = H_L(v, r) = \int_{S_r} v^2 \mu, \tag{2.1}$$

where μ is the *conformal factor*

$$\mu(x) = \mu_L(x) = \frac{\langle A(x)x, x \rangle}{|x|^2}.$$

We notice that, when $A(x) \equiv I$, then $\mu \equiv 1$. We also define the *generalized energy* of v in B_r

$$I(r) = I_L(v, r) = \int_{S_r} v \langle A \nabla v, \nu \rangle = D_L(v, r) + \int_{B_r} vf \tag{2.2}$$

where ν indicates the outer unit normal to S_r. The following result is Lemma 4.4 in [GSVG14].

Lemma 2.1. *The function $H(r)$ is absolutely continuous, and for a.e. $r \in (0, 1)$ one has*

$$H'(r) = 2I(r) + \int_{S_r} v^2 L|x|. \tag{2.3}$$

As it was explained in [GSVG14], the second term in the right-hand side of (2.3) above represents a serious difficulty to overcome if one wants to establish the monotonicity of the generalized frequency. To bypass this obstacle, one of the main ideas in [GSVG14] was the introduction of the following auxiliary functions, defined for v satisfying (1.12)–(1.15) and $0 < r < 1$:

$$\psi(r) = e^{\int_0^r G(s)ds}, \quad \sigma(r) = \frac{\psi(r)}{r^{n-2}}, \quad \text{where} \quad G(r) = \begin{cases} \frac{\int_{S_r} v^2 L|x|}{\int_{S_r} v^2 \mu}, & \text{if } H(r) \neq 0, \\ \frac{n-1}{r}, & \text{if } H(r) = 0. \end{cases} \tag{2.4}$$

When $L = \Delta$, it is easy to see that $\psi(r) = r^{n-1}$ and that $\sigma(r) = r$. We have the following simple and useful lemma which summarizes the most relevant properties of $\psi(r)$ and $\sigma(r)$.

Lemma 2.2. *There exists a universal constant $\beta \geq 0$ such that*

$$\frac{n-1}{r} - \beta \leq \frac{d}{dr} \log \psi(r) \leq \frac{n-1}{r} + \beta, \quad 0 < r < 1, \tag{2.5}$$

and one has

$$e^{-\beta(1-r)} r^{n-1} \leq \psi(r) \leq e^{\beta(1-r)} r^{n-1}, \quad 0 < r < 1. \tag{2.6}$$

This implies, in particular, $\psi(0+) = 0$. In terms of the function $\sigma(r) = \psi(r)/r^{n-2}$ we have

$$\left| \frac{d}{dr} \log \frac{\sigma(r)}{r} \right| \leq \beta, \quad 0 < r < 1 \tag{2.7}$$

and

$$e^{-\beta(1-r)} \leq \frac{\sigma(r)}{r} \leq e^{\beta(1-r)}, \quad 0 < r < 1. \tag{2.8}$$

In particular, $\sigma(0+) = 0$.

The next result is essentially Lemma 5.6 from [GSVG14].

Lemma 2.3. *There exist $\alpha > 0$ such that*

$$\left| \frac{\sigma(r)}{r} - \alpha \right| \leq \beta e^{\beta} r, \quad r \in (0, 1), \tag{2.9}$$

for β as in Lemma 2.2. In particular,

$$\alpha = \lim_{r \to 0+} \frac{\sigma(r)}{r}. \tag{2.10}$$

Moreover, we also have that $e^{-\beta} \leq \alpha \leq e^{\beta}$.

With ψ as in (2.4), we now define

$$M_L(v, r) = \frac{1}{\psi(r)} H_L(v, r), \quad J_L(v, r) = \frac{1}{\psi(r)} I_L(v, r). \tag{2.11}$$

The next relevant formulas are those of $J'(r) = \frac{d}{dr} J_L(v, r)$ and $M'(r) = \frac{d}{dr} M_L(v, r)$. Using formula (5.28) in [GSVG14], we have

$$\begin{aligned} J'(r) = \left(-\frac{\psi'(r)}{\psi(r)} + \frac{n-2}{r} + O(1) \right) J(r) + \frac{1}{\psi(r)} \bigg\{ 2 \int_{S_r} \frac{\langle A \nabla v, \nu \rangle^2}{\mu} \\ - \frac{2}{r} \int_{B_r} \langle Z, \nabla v \rangle f - \left(\frac{n-2}{r} + O(1) \right) \int_{B_r} v f + \int_{S_r} v f \bigg\}, \end{aligned} \tag{2.12}$$

where the vector field Z is given by

$$Z = \frac{r A \nabla r}{\mu} = \frac{A(x) x}{\mu}. \tag{2.13}$$

We also recall that (5.26) in [GSVG14] gives

$$M'(r) = 2J(r). \tag{2.14}$$

The central result in [GSVG14] is the following monotonicity formula.

Theorem 2.4 (Monotonicity of the truncated frequency)**.** *Let v satisfy* (1.11)–(1.15) *with $f \in L^\infty(B_1)$. Given $\delta \in (0,1)$ there exist universal numbers $r_0, K' > 0$, depending also on δ and $\|f\|_{L^\infty}$, such that the function*

$$N(r) = N_L(v,r) \overset{def}{=} \frac{\sigma(r)}{2} e^{K' r^{\frac{1-\delta}{2}}} \frac{d}{dr} \log \max\left\{M_L(v,r), r^{3+\delta}\right\}. \tag{2.15}$$

is monotone non-decreasing on $(0, r_0)$.

We call $N(r)$ the *truncated frequency function*, by analogy with Almgren's frequency function [Alm79] (see [GSVG14, GP09, CSS08] for more insights on this kind of formulas).

We then define a modification of N as follows:

$$\tilde N(r) = \tilde N_L(v,r) \overset{\text{def}}{=} \frac{r}{\sigma(r)} N_L(v,r) = \frac{r}{2} e^{K' r^{\frac{1-\delta}{2}}} \frac{d}{dr} \log \max\{M_L(v,r), r^{3+\delta}\}.$$

We notice that by Theorem 2.4 the limit $N(0+)$ exists. Combining that with Lemma 2.3 above, which states that $\lim\limits_{r\to 0+} \frac{\sigma(r)}{r} = \alpha > 0$, we see that $\tilde N(0+)$ also exists.

The following lemma provides a summary of estimates which are crucial for our further study. The lower bound on $\tilde N(0+)$ is proved in Lemma 6.3 in [GSVG14] (whose proof contains also that of the gap on the possible values of $\tilde N(0+)$), the bound on $|v(x)|$ is Lemma 6.6 and the bound on $|\nabla v(x)|$ is proved in Theorem 6.7 there.

Lemma 2.5. *Let v satisfy* (1.11)–(1.15) *with $f \in L^\infty(B_1)$, and let $r_0 \in (0, 1/2]$ be as in Theorem 2.4. Then, $\tilde N(0+) \geq \frac{3}{2}$, and actually $\tilde N(0+) = \frac{3}{2}$ or $\tilde N(0+) \geq \frac{3+\delta}{2}$.*

Moreover, there exists a universal C depending also on δ, $H(r_0)$ and $\|f\|_{L^\infty(B_1)}$ such that

$$|v(x)| \leq C|x|^{3/2}, \quad |\nabla v(x)| \leq C|x|^{1/2}, \quad |x| \leq r_0. \tag{2.16}$$

Corollary 2.6. *With r_0 as in Theorem 2.4, one has*

$$H(r) \leq C r^{n+2}, \quad |I(r)| \leq C r^{n+1}, \quad r \leq r_0. \tag{2.17}$$

Proof. It is enough to use (2.16) in definitions (2.1) and (2.2) above. ☐

The results of this section have been stated when the free boundary point in question is the origin. However, given any $x_0 \in \Gamma(v)$, we can move x_0 to the origin by letting

$$\begin{aligned}
v_{x_0}(x) &= v(x_0 + A^{1/2}(x_0)x) - b_{x_0} x_n, \quad \text{where } b_{x_0} = \langle A^{1/2}(x_0)\nabla v(x_0), e_n\rangle,\\
A_{x_0}(x) &= A^{-1/2}(x_0) A(x_0 + A^{1/2}(x_0)x) A^{-1/2}(x_0),\\
\mu_{x_0}(x) &= \langle A_{x_0}(x)\nu(x), \nu(x)\rangle,\\
L_{x_0} &= \operatorname{div}(A_{x_0}\nabla\cdot).
\end{aligned}$$

(Note that, by the $C^{1,\frac{1}{2}}$-regularity of v established in [GSVG14], the mapping $x_0 \mapsto b_{x_0}$ is $C^{\frac{1}{2}}$ on $\Gamma(v)$.) Then, by construction we have the normalizations $A_{x_0}(0) = I_n$, $\mu_{x_0}(0) = 1$. We also know that $0 \in \Gamma(v_{x_0})$, and that

$$v_{x_0}(0) = v(x_0) = 0, \quad |\nabla v_{x_0}(0)| = 0.$$

Besides, v_{x_0} satisfies (1.12)–(1.15) for the operator L_{x_0}. Thus, all results stated above for v are also applicable to v_{x_0}.

We thus also have the versions of the quantities defined in this sections, such as M_L, N_L, etc, centered at x_0 (if we replace L with L_{x_0}). But instead of using the overly bulky notations $M_{L_{x_0}}$, $N_{L_{x_0}}$, etc, we will use M_{x_0}, N_{x_0}, etc.

3. Regular free boundary points

Using Theorem 2.4, in this section we explore in more detail the notion of *regular free boundary points* and establish some preliminary properties of the regular set. We begin by recalling the following definition from Section 1.

Definition 3.1. We say that $x_0 \in \Gamma(v)$ is *regular* iff $\tilde{N}_{x_0}(v_{x_0}, 0+) = \frac{3}{2}$ and let $\Gamma_{3/2}(v)$ be the set of all regular free boundary points. $\Gamma_{3/2}(v)$ is also called the *regular set.*

In Lemma 3.3 below we prove that $\Gamma_{3/2}(v)$ is a relatively open subset of the free boundary $\Gamma(v)$. To accomplish this we prove that $\tilde{N}_{\bar{x}}(0+) = 3/2$ for $\bar{x}$ in a small neighborhood of $x_0 \in \Gamma_{3/2}(v)$. Since the definition of $\tilde{N}(r)$ involves a truncation of $M(r)$, we first need to establish the following auxiliary result.

Lemma 3.2. *Let v satisfy* (1.12)–(1.15) *with $0 \in \Gamma_{3/2}(v)$. Then*

$$\frac{r}{2}\frac{M'(r)}{M(r)} \to \frac{3}{2} \quad \text{as } r \to 0+\,.$$

In particular, for every $\varepsilon > 0$ there exists $r_\varepsilon > 0$ and $C_\varepsilon > 0$ such that

$$\frac{r}{2}\frac{M'(r)}{M(r)} \le \frac{3+\varepsilon}{2}, \quad M(r) \ge C_\varepsilon r^{3+\varepsilon}, \quad \text{for } 0 < r \le r_\varepsilon.$$

Proof. We first claim that since $0 \in \Gamma_{3/2}(v)$, then $M(r) \ge r^{3+\delta}$ for $r > 0$ small. Indeed, if there was a sequence $s_j \to 0$ such that $M(s_j) < s_j^{3+\delta}$, then

$$\tilde{N}(s_j) = \frac{3+\delta}{2} e^{K' s_j^{\frac{1-\delta}{2}}} \to \frac{3+\delta}{2} \ne \frac{3}{2} = \tilde{N}(0+),$$

which is a contradiction. Hence, for r small we have

$$\tilde{N}(r) = \frac{r}{2} e^{K' r^{\frac{1-\delta}{2}}} \frac{M'(r)}{M(r)}.$$

Since $e^{K' r^{\frac{1-\delta}{2}}} \to 1$ as $r \to 0$ and $\tilde{N}(0+) = 3/2$, we obtain the first part of the lemma. Hence, for every $\varepsilon > 0$ there exists a small $r_\varepsilon > 0$ such that

$$r\frac{M'(r)}{M(r)} \le 3+\varepsilon, \quad r < r_\varepsilon.$$

Integrating from r to r_ε, this gives

$$\frac{M(r_\varepsilon)}{M(r)} \le \left(\frac{r_\varepsilon}{r}\right)^{3+\varepsilon},$$

from which we conclude, with $C_\varepsilon = M(r_\varepsilon)/r_\varepsilon^{3+\varepsilon}$, that $M(r) \ge C_\varepsilon r^{3+\varepsilon}$. □

Lemma 3.3. *Let v satisfy* (1.12)–(1.15) *with $x_0 \in \Gamma_{3/2}(v)$. Then, there exists $\eta_0 = \eta_0(x_0) > 0$ such that $\Gamma(v) \cap B'_{\eta_0}(x_0) \subset \Gamma_{3/2}(v)$ and, moreover, the convergence*

$$\tilde{N}_{\bar{x}}(r) \to 3/2 \quad \text{as} \quad r \to 0+$$

is uniform for $\bar{x} \in \Gamma(v) \cap \overline{B'_{\eta_0/2}(x_0)}$.

Proof. Fix $0<\varepsilon<\delta/8$, where $\delta>0$ is fixed as in the definition (2.15) of the frequency $N_{x_0}(r)$. Then, by Lemma 3.2, there exist $C_\varepsilon=C_\varepsilon(x_0)>0$ and $r_\varepsilon=r_\varepsilon(x_0)>0$ such that

$$\frac{r}{2}\frac{M_{x_0}'(r)}{M_{x_0}(r)}\leq\frac{3+\varepsilon}{2},\quad M_{x_0}(r)\geq C_\varepsilon r^{3+\varepsilon},\quad r<r_\varepsilon.$$

We then want to show that similar inequalities will hold if we replace x_0 with $\bar{x}\in B_{\eta_\varepsilon}'(x_0)$ for a sufficiently small η_ε. We will write $L_{\bar{x}}=\operatorname{div}(A_{\bar{x}}\nabla\cdot)$. To track the dependence on $\bar{x}$, we write, using the differentiation formulas in [GSVG14], that

$$\begin{aligned}\frac{M_{\bar{x}}'(r)}{M_{\bar{x}}(r)}&=\frac{H_{\bar{x}}'(r)}{H_{\bar{x}}(r)}-\frac{\psi_{\bar{x}}'(r)}{\psi_{\bar{x}}(r)}=\frac{H_{\bar{x}}'(r)}{H_{\bar{x}}(r)}-\frac{\int_{S_r}v_{\bar{x}}(x)^2L_{\bar{x}}|x|}{\int_{S_r}v_{\bar{x}}(x)^2\mu_{\bar{x}}(x)}\\&=\frac{2I_{\bar{x}}(r)}{H_{\bar{x}}(r)}=\frac{2\int_{S_r}v_{\bar{x}}(x)\langle A_{\bar{x}}(x)\nabla v_{\bar{x}}(x),\nu\rangle}{\int_{S_r}v_{\bar{x}}(x)^2\mu_{\bar{x}}(x)}\\&=\frac{2r\int_{S_r}v_{\bar{x}}(x)\langle A_{\bar{x}}(x)\nabla v_{\bar{x}}(x),x\rangle}{\int_{S_r}v_{\bar{x}}(x)^2\langle A_{\bar{x}}(x)x,x\rangle}.\end{aligned}$$

This implies that for fixed $r>0$, the mapping $\bar{x}\mapsto M_{\bar{x}}'(r)/M_{\bar{x}}(r)$ is continuous. Thus, if $\rho_\varepsilon<r_\varepsilon$, we can find a small $\eta_\varepsilon>0$ such that

$$\frac{\rho_\varepsilon}{2}\frac{M_{\bar{x}}'(\rho_\varepsilon)}{M_{\bar{x}}(\rho_\varepsilon)}\leq\frac{3+2\varepsilon}{2},\quad\text{for }\bar{x}\in B_{\eta_\varepsilon}'(x_0).$$

On the other hand, since

$$M_{\bar{x}}(r)=\frac{r}{\sigma_{\bar{x}}(r)}\frac{1}{r^{n+1}}\int_{S_r}v_{\bar{x}}(x)^2\langle A_{\bar{x}}(x)x,x\rangle,$$

the continuity of the mapping $\bar{x}\to M_{\bar{x}}(r)$ is not so clear. However, having that $c_0\leq\frac{\sigma_{\bar{x}}(r)}{r}\leq c_0^{-1}$ for a universal constant $c_0>0$, we can write that

$$M_{\bar{x}}(\rho_\varepsilon)\geq\frac{1}{2}c_0^2C_\varepsilon\rho_\varepsilon^{3+\varepsilon}>\rho_\varepsilon^{3+\delta},\quad\bar{x}\in B_{\eta_\varepsilon}'(x_0),$$

if we take η_ε and ρ_ε sufficiently small. The latter inequality implies that we can explicitly compute $\tilde{N}_{\bar{x}}(\rho_\varepsilon)$ by

$$\tilde{N}_{\bar{x}}(\rho_\varepsilon)=\frac{\rho_\varepsilon}{2}e^{K'\rho_\varepsilon^{(1-\delta)/2}}\frac{M_{\bar{x}}'(\rho_\varepsilon)}{M_{\bar{x}}(\rho_\varepsilon)}\leq e^{K'\rho_\varepsilon^{(1-\delta)/2}}\frac{3+2\varepsilon}{2}\leq\frac{3+3\varepsilon}{2},$$

again if ρ_ε is small enough. Hence, by the monotonicity of $N_{\bar{x}}(r)=(\sigma_{\bar{x}}(r)/r)\tilde{N}_{\bar{x}}(r)$, we obtain

$$\tilde{N}_{\bar{x}}(0+)\leq\frac{1}{\alpha_{\bar{x}}}\frac{\sigma_{\bar{x}}(\rho_\varepsilon)}{\rho_\varepsilon}\frac{3+3\varepsilon}{2},$$

where

$$\alpha_{\bar{x}}\overset{\text{def}}{=}\lim_{r\to0+}\frac{\sigma_{\bar{x}}(r)}{r}.\tag{3.1}$$

Using now the estimates in Lemma 2.3, we have

$$\left|\frac{\sigma_{\bar{x}}(r)}{r}-\alpha_{\bar{x}}\right|\leq C_0r\quad\text{and}\quad\alpha_{\bar{x}}\geq c_0,$$

therefore we can guarantee that

$$\tilde{N}_{\bar{x}}(0+)\leq(1+\varepsilon)\frac{3+3\varepsilon}{2}\leq\frac{3+7\varepsilon}{2}<\frac{3+\delta}{2},\quad\bar{x}\in B_{\eta_\varepsilon}'(x_0).$$

But then, by the gap of values of $\tilde N_{\bar x}(0+)$ between $3/2$ and $(3+\delta)/2$, we conclude that

$$\tilde N_{\bar x}(0+) = 3/2, \quad \bar x \in B'_{\eta_\varepsilon}(x_0).$$

To prove the second part of the lemma, we note that for any fixed $\bar x \in B'_{\eta_\varepsilon}(x_0)$, the mapping

$$r \mapsto e^{\beta r}\tilde N_{\bar x}(r) \quad 0 < r < \rho_\varepsilon$$

is monotone increasing for a universal constant $\beta > 0$, which follows from the inequality

$$\left|\frac{d}{dr}\log\frac{\sigma_{\bar x}(r)}{r}\right| \le \beta, \qquad \text{(see Lemma 2.3)},$$

the monotonicity of $r \mapsto N_{\bar x}(r) = \frac{\sigma_{\bar x}(r)}{r}\tilde N_{\bar x}(r)$, as well as the nonnegativity of $\tilde N_{\bar x}(r)$.

Now, for each fixed $0 < r < \rho_\varepsilon$, the mapping

$$\bar x \mapsto e^{\beta r}\tilde N_{\bar x}(r) = e^{\beta r + K' r^{(1-\delta)/2}}\frac{r^2\int_{S_r} v_{\bar x}(x)\langle A_{\bar x}(x)\nabla v_{\bar x}(x), x\rangle}{\int_{S_r} v_{\bar x}(x)^2\langle A_{\bar x}(x)x, x\rangle}$$

is continuous on $B'_{\eta_\varepsilon}(x_0)$. Since the limit

$$\lim_{r\to 0+} e^{\beta r}\tilde N_{\bar x}(r) = \tilde N_{\bar x}(0+) = 3/2, \quad \text{for all } \bar x \in B'_{\eta_\varepsilon}(x_0),$$

by the classical theorem of Dini, we have that the convergence $e^{\beta r}\tilde N_{\bar x}(r) \to 3/2$ as $r \to 0+$ will be uniform on $\Gamma(v) \cap \overline{B'_{\eta_\varepsilon/2}(x_0)}$, implying also the uniform convergence $\tilde N_{\bar x}(r) \to 3/2$. $\square$

In the remaining part of this section we study *Almgren type scalings*

$$\tilde v_{\bar x,r}(x) = \frac{v_{\bar x}(rx)}{d_{\bar x,r}}, \quad d_{\bar x,r} = \left(\frac{1}{r^{n-1}}H_{\bar x}(r)\right)^{1/2}. \tag{3.2}$$

This is slightly different from what was done in [GSVG14], but more suited for the study of the free boundary. Notice that we have the following normalization:

$$\int_{S_1}\tilde v_{\bar x,r}^2\mu_{\bar x,r} = 1, \quad \mu_{\bar x,r} = \frac{\langle A_{\bar x}(rx)x, x\rangle}{|x|^2}.$$

Now, if $\bar x \in \Gamma_{3/2}(v)$, then the results of [GSVG14] imply that over subsequences $r = r_j \to 0$, we have the convergence

$$\tilde v_{\bar x,r}(x)\left(\frac{\sigma_{\bar x}(r)}{r}\right)^{1/2} \to a\Re(\langle x', e'\rangle + i|x_n|)^{3/2} \quad \text{in } C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n_\pm \cup \mathbb{R}^{n-1})$$

for some $e' \in \mathbb{R}^{n-1}$ with $|e'| = 1$ and $a > 0$. Since we also have the convergence

$$\frac{\sigma_{\bar x}(r)}{r} \to \alpha_{\bar x} > 0,$$

see Lemma 2.3, we obtain that

$$\tilde v_{\bar x,r}(x) \to \frac{a}{\alpha_{\bar x}^{1/2}}\Re(\langle x', e'\rangle + i|x_n|)^{3/2} \quad \text{in } C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n_\pm \cup \mathbb{R}^{n-1}).$$

The normalization $\int_{S_1}\tilde v_{\bar x,r}^2\mu_{\bar x,r} = 1$ then implies that there exists a dimensional constant $c_n > 0$ (independent of $\bar x$), such that, on a subsequence,

$$\tilde v_{\bar x,r}(x) \to c_n\Re(\langle x', e'\rangle + i|x_n|)^{3/2}.$$

Moreover, we can actually prove the following result.

Lemma 3.4. *Let v satisfy* (1.12)–(1.15) *with $x_0 \in \Gamma_{3/2}(v)$. Given $\theta > 0$, there exists $r_0 = r_0(x_0) > 0$ and $\eta_0 = \eta_0(x_0) > 0$ such that*

$$\inf_{e' \in \mathbb{R}^{n-1}, |e'|=1} \|\tilde{v}_{\bar{x},r}(x) - c_n \Re(\langle x', e' \rangle + i|x_n|)^{3/2}\|_{C^{1,\alpha}(B_1^\pm \cup B_1')} < \theta$$

for any $\bar{x} \in = \Gamma_{3/2}(v) \cap \overline{B'_{\eta_0}(x_0)}$ and $r < r_0$.

Proof. Suppose the contrary. Then, there exists a sequence $\bar{x}_j \to x_0$ and $r_j \to 0$ such that

$$\|\tilde{v}_{\bar{x}_j,r_j} - c_n \Re(\langle x', e' \rangle + i|x_n|)^{3/2}\|_{C^{1,\alpha}(B_1^\pm \cup B_1')} \geq \theta$$

for any unit vector $e' \in \mathbb{R}^{n-1}$. Observe now that

$$\begin{aligned} e^{-K'(\rho r)^{(1-\delta)/2}} \tilde{N}_{\bar{x}}(\rho r) &= \frac{(\rho r)^2 \int_{S_{\rho r}} v_{\bar{x}}(x) \langle A_{\bar{x}}(x) \nabla v_{\bar{x}}(x), x \rangle}{\int_{S_{\rho r}} v_{\bar{x}}(x)^2 \langle A_{\bar{x}}(x) x, x \rangle} \\ &= \frac{\rho^2 \int_{S_\rho} \tilde{v}_{\bar{x},r}(x) \langle A_{\bar{x}}(rx) \nabla \tilde{v}_{\bar{x},r}(x), x \rangle}{\int_{S_\rho} \tilde{v}_{\bar{x},r}(x)^2 \langle A_{\bar{x}}(rx) x, x \rangle} \end{aligned} \tag{3.3}$$

Now, we claim that the scalings $\tilde{v}_{\bar{x}_j,r_j}$ are uniformly bounded in $C^{1,1/2}(B_R^\pm \cup B_R')$ for any $R > 0$. Indeed, by Lemma 3.3, given $\varepsilon > 0$ small, we have that

$$\frac{t}{2} \frac{M_{\bar{x}}'(t)}{M_{\bar{x}}(t)} \leq \frac{3 + \varepsilon}{2}, \quad t < \rho_\varepsilon,\ \bar{x} \in B'_{\eta_\varepsilon}(x_0).$$

Let $R \geq 1$ and $Rr < \rho_\varepsilon$. Integrating the above inequality from $t = r$ to Rr, we obtain that

$$M_{\bar{x}}(Rr) \leq M_{\bar{x}}(r) R^{3+\varepsilon}.$$

Changing $M_{\bar{x}}$ to $H_{\bar{x}}$ we therefore have, using (2.6), that

$$H_{\bar{x}}(Rr) \leq C_0 H_{\bar{x}}(r) R^{n+2+\varepsilon}.$$

The latter can we written in the form

$$\int_{S_R} \tilde{v}_{\bar{x},r}^2 \mu_{\bar{x},r} \leq C_0 R^{n+2+\varepsilon}.$$

Thus, the uniform boundedness of $\tilde{v}_{\bar{x}_j,r_j}$ in $L^2(B_R)$, and consequently in $C^{1,1/2}(B_{R/2}^\pm \cup B_{R/2}')$, follows. Hence, we can assume without loss of generality that $\tilde{v}_{\bar{x}_j,r_j} \to v_0$ in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n_\pm \cup \mathbb{R}^{n-1})$. It is immediate to see that v_0 will solve the Signorini problem for the Laplacian in the entire $\mathbb{R}^n$. Besides, by Lemma 3.3, we will have that $\tilde{N}_{\bar{x}_j}(\rho r_j) \to 3/2$. On the other hand, passing to the limit in (3.3), we obtain that

$$\frac{3}{2} = \frac{\int_{S_\rho} v_0(x) \langle \nabla v_0(x), x \rangle}{\int_{S_\rho} v_0(x)^2} = \frac{\rho \int_{B_\rho} |\nabla v_0|^2}{\int_{S_\rho} v_0(x)^2}, \quad \text{for any } \rho > 0.$$

Therefore, v_0 is $3/2$ homogeneous global solution of the Signorini problem and thus has the form $v_0(x) = c_n \Re(\langle x', e_0' \rangle + i|x_n|)^{3/2}$. Hence, for large j we will have

$$\|\tilde{v}_{\bar{x}_j,r_j} - c_n \Re(\langle x', e_0' \rangle + i|x_n|)^{3/2}\|_{C^1(B_1^\pm \cup B_1')} < \theta,$$

contradictory to our assumption. The proof is complete. □

4. A Weiss type monotonicity formula

In this section we establish a monotonicity formula which is reminiscent of that established by Weiss in [Wei99] for the classical obstacle problem, and which is one of the two main ingredients in our proof of the $C^{1,\beta}$ regularity of the regular set. We consider the solution u to the Signorini problem (1.6)–(1.9) above, and we set v as in (1.10).

Definition 4.1. Let $r_0 > 0$ be as in Theorem 2.4. For $r \in (0, r_0)$ we define the $\frac{3}{2}$*-th generalized Weiss-type functional* as follows

$$W_L(v,r) = \frac{\sigma(r)}{r^3}\left\{J_L(v,r) - \frac{3/2}{r}M_L(v,r)\right\} \tag{4.1}$$

$$= \frac{1}{r^{n+1}}I_L(v,r) - \frac{3/2}{r^{n+2}}H_L(v,r) \tag{4.2}$$

$$= \frac{1}{r^{n+1}}\int_{B_r}[\langle A(x)\nabla v, \nabla v\rangle + vf] - \frac{3/2}{r^{n+2}}\int_{S_r} v^2\mu,$$

where I_L, J_L, H_L, and $M_L(v,r)$ are as in Section 2. Whenever $L = \Delta$, we write $W(v,r) = W_\Delta(v,r)$, and unless we want to stress the dependence on v, we will write $W_L(r) = W_L(v,r)$.

In this section we will show that there exists $C > 0$ such that $r \mapsto W_L(v,r) + Cr^{1/2}$ is monotone nondecreasing, that the limit $\lim_{r\to 0} W_L(v,r)$ exists and is zero, and that $W_L(v,r) \geq -Cr^{1,2}$. We start by proving that $W_L(v,\cdot)$ is bounded.

Lemma 4.2. *The functional $W_L(v,\cdot)$ is bounded on the interval $(0, r_0)$.*

Proof. Indeed, from Corollary 2.6 we have

$$|W_L(v,r)| \leq \frac{1}{r^{n+1}}\left\{|I(r)| + \frac{3}{2r}H(r)\right\} \leq C. \qquad \square$$

The functional $W_L(r)$ in (4.1) is tailor-made to the study of regular free boundary points of solutions of the Signorini problem (1.12)–(1.15). The following "almost monotonicity" property of W_L plays a crucial role in our further study.

Theorem 4.3 (Weiss type monotonicity formula)**.** *Let v satisfy* (1.12)*–*(1.15) *and let $0 \in \Gamma_{3/2}(v)$. Then, there exist universal constant $C, r_0 > 0$, depending also on $\|f\|_{L^\infty(B_1)}$, such that for every $0 < r < r_0$ one has*

$$\frac{d}{dr}\left(W_L(v,r) + Cr^{1/2}\right) \geq \frac{2}{r^{n+1}}\int_{S_r}\left(\frac{\langle A\nabla v, \nu\rangle}{\sqrt{\mu}} - \frac{(3/2)\sqrt{\mu}}{r}v\right)^2. \tag{4.3}$$

In particular, there exists $C > 0$ such that function $r \mapsto W_L(v,r) + Cr^{1/2}$ is monotone nondecreasing, and therefore the limit $W_L(v,0+) \overset{def}{=} \lim_{r\to 0} W_L(v,r)$ exists.

Proof. We have from Definition 4.1 above,

$$\frac{d}{dr}W_L(v,r) = \frac{\sigma(r)}{r^3}\left\{\left(\frac{\sigma'(r)}{\sigma(r)} - \frac{3}{r}\right)\left(J(r) - \frac{3/2}{r}M(r)\right) + \left(J'(r) - \frac{3/2}{r}M'(r) + \frac{3/2}{r^2}M(r)\right)\right\}$$

$$= \frac{\sigma(r)}{r^3}\left\{\left(\frac{\psi'(r)}{\psi(r)} - \frac{n-2}{r} - \frac{3}{r}\right)\left(J(r) - \frac{3/2}{r}M(r)\right) + \left(J'(r) - \frac{3/2}{r}M'(r) + \frac{3/2}{r^2}M(r)\right)\right\}.$$

Using (2.12) and (2.14) above, we thus find

$$
\begin{aligned}
\frac{d}{dr}W_L(v,r) &= \frac{\sigma(r)}{r^3}\Big\{\Big(\frac{\psi'(r)}{\psi(r)} - \frac{n-2}{r} - \frac{3}{r}\Big)\Big(J(r) - \frac{3/2}{r}M(r)\Big) \\
&\quad + \Big(\Big(-\frac{\psi'(r)}{\psi(r)} + \frac{n-2}{r} + O(1)\Big)J(r) + \frac{1}{\psi(r)}\Big\{2\int_{S_r}\frac{\langle A\nabla v,\nu\rangle^2}{\mu} \\
&\quad - \frac{2}{r}\int_{B_r}\langle Z,\nabla v\rangle f - \Big(\frac{n-2}{r} + O(1)\Big)\int_{B_r} vf + \int_{S_r} vf\Big\} - \frac{3}{r}J(r) + \frac{3/2}{r^2}M(r)\Big)\Big\} \\
&= \frac{\sigma(r)}{r^3}\Big\{\Big(-\frac{6}{r} + O(1)\Big)J(r) + \frac{2}{\psi(r)}\int_{S_r}\frac{\langle A\nabla v,\nu\rangle^2}{\mu} \\
&\quad + \frac{3/2}{r^2}\Big(1 - \Big(\frac{r\psi'(r)}{\psi(r)} - n + 2 - 3\Big)\Big)M(r) \\
&\quad + \frac{1}{\psi(r)}\Big(-\frac{2}{r}\int_{B_r}\langle Z,\nabla v\rangle f - \Big(\frac{n-2}{r} + O(1)\Big)\int_{B_r} vf + \int_{S_r} vf\Big)\Big\}.
\end{aligned}
$$

By (2.5) in Lemma 2.2 above, we see that $\frac{\psi'(r)}{\psi(r)} = \frac{n-1}{r} + O(1)$. We thus obtain from the latter chain of equalities

$$
\begin{aligned}
\frac{d}{dr}W_L(v,r) &= \frac{2\sigma(r)}{r^3\psi(r)}\Big\{\Big(-\frac{3}{r} + O(1)\Big)I(r) + \int_{S_r}\frac{\langle A\nabla v,\nu\rangle^2}{\mu} + \frac{9/4}{r^2}\big(1 + O(r)\big)H(r)\Big\} \\
&\quad + \frac{\sigma(r)}{r^3\psi(r)}\Big(-\frac{2}{r}\int_{B_r}\langle Z,\nabla v\rangle f - \Big(\frac{n-2}{r} + O(1)\Big)\int_{B_r} vf + \int_{S_r} vf\Big).
\end{aligned}
$$

By the definitions (2.1) and (2.2) of $H(r)$ and $I(r)$ we have

$$
\int_{S_r}\Big(\frac{\langle A\nabla v,\nu\rangle}{\sqrt{\mu}} - \frac{(3/2)\sqrt{\mu}}{r}v\Big)^2 = \int_{S_r}\frac{\langle A\nabla v,\nu\rangle^2}{\mu} - \frac{3}{r}I(r) + \frac{9/4}{r^2}H(r).
$$

Since $\sigma(r) = r^{2-n}\psi(r)$, we conclude that

$$
\begin{aligned}
\frac{d}{dr}W_L(v,r) &= \frac{2}{r^{n+1}}\Big\{\int_{S_r}\Big(\frac{\langle A\nabla v,\nu\rangle}{\sqrt{\mu}} - \frac{(3/2)\sqrt{\mu}}{r}v\Big)^2 + O(1)I(r) + \frac{O(1)}{r}H(r)\Big\} \\
&\quad + \frac{1}{r^{n+1}}\Big(-\frac{2}{r}\int_{B_r}\langle Z,\nabla v\rangle f - \Big(\frac{n-2}{r} + O(1)\Big)\int_{B_r} vf + \int_{S_r} vf\Big).
\end{aligned}
\tag{4.4}
$$

Returning to (4.4) and making use of (2.17), we conclude that

$$
\begin{aligned}
\frac{d}{dr}W_L(v,r) &= \frac{2}{r^{n+1}}\int_{S_r}\Big(\frac{\langle A\nabla v,\nu\rangle}{\sqrt{\mu}} - \frac{(3/2)\sqrt{\mu}}{r}v\Big)^2 + O(1) \\
&\quad + \frac{1}{r^{n+1}}\Big(-\frac{2}{r}\int_{B_r}\langle Z,\nabla v\rangle f - \Big(\frac{n-2}{r} + O(1)\Big)\int_{B_r} vf + \int_{S_r} vf\Big).
\end{aligned}
\tag{4.5}
$$

The proof of the estimate (4.3) will be completed if we can show that there exists a universal constant $C > 0$ such that

$$
\Big| -\frac{2}{r}\int_{B_r}\langle Z,\nabla v\rangle f - \Big(\frac{n-2}{r} + O(1)\Big)\int_{B_r} vf + \int_{S_r} vf\Big| \le C r^{n+\frac{1}{2}}.
$$

From the expression of the vector field $Z = \frac{A(x)x}{\mu}$, see (2.13) above, we have $|Z| \leq Cr$ for $|x| \leq r$. Since $f \in L^\infty$, we obtain from the second inequality in (2.16)

$$\left| -\frac{2}{r} \int_{B_r} \langle Z, \nabla v \rangle f \right| \leq C r^{n+\frac{1}{2}},$$

for a universal $C > 0$ which also depends on $\|f\|_{L^\infty(B_1)}$. The first inequality in (2.16) gives instead

$$\left| \left(\frac{n-2}{r} + O(1) \right) \int_{B_r} vf + \int_{S_r} vf \right| \leq C r^{n+\frac{1}{2}},$$

thus completing the proof of (4.3). The existence of $W_L(v, 0+)$ now follows from the monotonicity and boundedness of $W_L(v, r) + Cr^{1/2}$, see Lemma 4.2. □

From Theorem 4.3 we obtain that $W_L(v, 0+) = \lim_{r \to 0} W_L(v, r)$ exists. In the next lemma we prove that this limit must actually be zero.

Lemma 4.4. *Let v satisfy (1.11)–(1.15) with $f \in L^\infty(B_1)$ and $0 \in \Gamma_{3/2}(v)$. Then, $W_L(v, 0+) = 0$.*

Proof. Recall from (4.1) that one has

$$W_L(v, r) = W_L(r) = \frac{\sigma(r)}{2r^3} \left\{ 2J(r) - \frac{3}{r} M(r) \right\} = \frac{H(r)}{r^{n+2}} \left\{ \frac{r}{2} \frac{M'(r)}{M(r)} - \frac{3}{2} \right\}, \tag{4.6}$$

where in the last equality we have used (2.14) and (2.11) above. The proof now follows from the boundedness of $\frac{H(r)}{r^{n+2}}$ by (2.17) and the convergence $\frac{r}{2} \frac{M'(r)}{M(r)} \to \frac{3}{2}$ by Lemma 3.2. □

Corollary 4.5. *Let C and r_0 be as in Theorem 4.3. Then, for every $0 < r < r_0$ one has*

$$W_L(v, r) \geq -Cr^{1/2}.$$

Proof. This follows directly by combining Theorem 4.3 with Lemma 4.4. □

5. Homogeneous blowups

In this section we analyze the uniform limits of some appropriate scalings of a solution v to the Signorini problem (1.12)–(1.15) by making essential use of the monotonicity formula in Theorem 4.3 above. These scalings, together with the Almgren type ones defined in (3.2), will be instrumental in Section 7.

Let v satisfy (1.12)–(1.15) and let $0 \in \Gamma_{3/2}(v)$. We consider the following *homogeneous scalings* of v

$$v_r(x) = \frac{v(rx)}{r^{3/2}}. \tag{5.1}$$

Lemma 5.1. *Define $A_r(x) = A(rx)$ and $f_r(x) = r^{1/2} f(rx)$. Then, v_r solves the thin obstacle problem (1.12)–(1.16) in $B_{\frac{1}{r}}$ with zero thin obstacle, operator $L_r = \operatorname{div}(A_r \nabla \cdot)$ and right-hand side f_r in (1.12).*

Proof. We only need to verify (1.16). Indeed, given $\eta \in C_0^\infty(B_{\frac{1}{r}})$, define $\rho \in C_0^\infty(B_1)$ by letting $\rho(y) = r^{1/2}\eta\left(\frac{y}{r}\right)$. By a change of variable one easily verifies that

$$\int_{B_{\frac{1}{r}}} \langle A_r \nabla v_r, \nabla \eta\rangle = \int_{B'_{\frac{1}{r}}} \left[\langle A_r(x)\nabla v_r(x), \nu_+\rangle + \langle A_r(x)\nabla v_r(x), \nu_-\rangle\right]\eta(x) - \int_{B_{\frac{1}{r}}} f_r(x)\eta(x). \tag{5.2}$$

□

Lemma 5.2. *Let v satisfy* (1.12)–(1.15) *and let* $0 \in \Gamma_{3/2}(v)$. *Given $r_j \to 0$, there exists a subsequence (which we will still denote by r_j) and a function $v_0 \in C^{1,\alpha}_{loc}(\mathbb{R}^n_\pm \cup \mathbb{R}^{n-1})$ for any $\alpha \in (0, 1/2)$, such that $v_{r_j} \to v_0$ in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n_\pm \cup \mathbb{R}^{n-1})$. Such v_0 is a global solution of the Signorini problem* (1.12)–(1.15) *in $\mathbb{R}^n$ with zero thin obstacle and zero right-hand side f.*

Proof. By Lemma 2.5, there exist universal constants $C, r_0 > 0$ such that

$$|v(x)| \le C|x|^{3/2} \quad \text{and} \quad |\nabla v(x)| \le C|x|^{1/2}, \quad |x| < r_0.$$

Moreover, as proved in [GSVG14], $v \in C^{1,\frac{1}{2}}_{\text{loc}}(B_1^\pm \cup B_1')$ with

$$\|v\|_{C^{1,\frac{1}{2}}(B^\pm_{\frac{1}{2}} \cup B'_{\frac{1}{2}})} \le C(n, \lambda, Q, \|v\|_{W^{1,2}(B_1)}, \|f\|_{L^\infty}).$$

Given $r_j \searrow 0$, by a standard diagonal process we obtain convergence in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n_\pm \cup \mathbb{R}^{n-1})$, for any $\alpha \in (0, 1/2)$, of a subsequence of the functions v_{r_j} to a function v_0. Passing to the limit in (5.2) we conclude that v_0 is a global solution to the Signorini problem with zero thin obstacle. The fact (important for our later purposes) that the right-hand side f is also zero, follows again from (5.2) since we obtain

$$\left|\int_{B_{\frac{1}{r}}} f_r(x)\eta(x)\right| = \sqrt{r}\left|\int_{B_{\frac{1}{r}}} f(rx)\eta(x)\right| \le \|f\|_{L^\infty(B_1)}\|\eta\|_{L^1(\mathbb{R}^n)}\sqrt{r} \longrightarrow 0.$$

□

Remark 5.3. Notice that we have not yet ruled out the possibility that $v_0 \equiv 0$. This crucial aspect will be dealt with later, in Proposition 7.3 below. The next result establishes an important homogeneity property of the global solution v_0.

Proposition 5.4. *Let v_0 be a function as in Lemma 5.2. Then, v_0 is homogeneous of degree* 3/2.

Proof. In what follows we denote with $\mu_r(x) = \mu(rx)$. Furthermore, recall that the scaled functions in (5.1) are given by $v_r(x) = r^{-3/2}\, v(rx)$. Let $r_j \searrow 0$ and denote by v_0 a corresponding blowup as

in Lemma 5.2. Let $0 < r < R$. We integrate (4.3) over the interval (r, R) obtaining

$$
\begin{aligned}
W_L(v, r_jR) - W_L(v, r_jr) - C\sqrt{r_j}(\sqrt{R} - \sqrt{r}) &\geq \int_{r_jr}^{r_jR} \frac{2}{t^{n+3}} \int_{S_t} \left(\frac{\langle A\nabla v, x\rangle}{\sqrt{\mu}} - \frac{3}{2}\sqrt{\mu} v \right)^2 dt \\
&= \int_r^R \frac{1}{(r_js)^{n+3}} \int_{S_{r_js}} \left(\frac{\langle A\nabla v, x\rangle}{\sqrt{\mu}} - \frac{3}{2}\sqrt{\mu} v \right)^2 r_j ds \\
&= \int_r^R \frac{1}{(r_js)^{n+3}} \int_{S_s} \left(\frac{\langle A(r_jy)\nabla v(r_jy), r_jy\rangle}{\sqrt{\mu(r_jy)}} - \frac{3}{2}\sqrt{\mu(r_jy)} v(r_jy) \right)^2 r_j^{n-1} r_j ds \\
&= \frac{1}{r_j^3} \int_r^R \frac{1}{s^{n+3}} \int_{S_s} \left(\frac{\langle A_{r_j}(y)\nabla v(r_jy), y\rangle r_j}{\sqrt{\mu_{r_j}(y)}} - \frac{3}{2}\sqrt{\mu_{r_j}(y)} v(r_jy) \right)^2 ds \\
&= \frac{1}{r_j^3} \int_r^R \frac{1}{s^{n+3}} \int_{S_s} \left(\frac{\langle A_{r_j}(y) r_j^{1/2} \nabla v_{r_j}(y), y\rangle r_j}{\sqrt{\mu_{r_j}(y)}} - \frac{3}{2}\sqrt{\mu_{r_j}(y)} v_{r_j}(y) r^{3/2} \right)^2 ds \\
&= \int_r^R \frac{1}{s^{n+3}} \int_{S_s} \left(\frac{\langle A_{r_j}(y)\nabla v_{r_j}(y), y\rangle}{\sqrt{\mu_{r_j}(y)}} - \frac{3}{2}\sqrt{\mu_{r_j}(y)} v_{r_j}(y) \right)^2 ds.
\end{aligned}
$$

Since $W_L(v, 0+)$ exists, the left-hand side goes to zero as $j \to \infty$. Since $A(0) = I, \mu(0) = 1$ and we have $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n_\pm \cup \mathbb{R}^{n-1})$ convergence of v_{r_j} to v_0, passing to the limit as $j \to \infty$ in the above, we conclude that

$$
0 \geq \int_r^R \frac{1}{s^{n+3}} \int_{S_s} \left(\langle \nabla v_0(y), y\rangle - \frac{3}{2} v_0(y) \right)^2 ds.
$$

This inequality, and the arbitrariness of $0 < r < R$, imply that v_0 is homogeneous of degree $3/2$. $\square$

Definition 5.5. We call such v_0 a *homogeneous blowup.*

6. An epiperimetric inequality for the Signorini problem

In this section we establish in the context of the Signorini problem a basic generalization of the epiperimetric inequality obtained by Weiss for the classical obstacle problem. Our main result, which is Theorem 6.3 below, is tailor made for analyzing regular free boundary points in the Signorini problem, being the second main tool we use to reach our goal – the $C^{1,\beta}$ regularity of the regular set.

Definition 6.1. Given $v \in W^{1,2}(B_1)$, we define the *boundary adjusted energy* as the Weiss type functional defined in (4.1) for the Laplacian operator with $r = 1$ and zero thin obstacle, i.e.,

$$
W(v) \stackrel{\mathrm{def}}{=} W_\Delta(v, 1) = \int_{B_1} |\nabla v|^2 - \frac{3}{2} \int_{S_1} v^2.
$$

Remark 6.2. We observe explicitly that if $\int_{S_1} v^2 \neq 0$, then we can write

$$
W(v) = \left(\int_{S_1} v^2 \right) \left[N(v, 1) - \frac{3}{2} \right].
$$

It follows that if v is a solution to the Signorini for the Laplacian in $\mathbb{R}^n$, with zero thin obstacle, and which is homogeneous of degree $\frac{3}{2}$, then by [ACS08] we have $N(v, r) \equiv \frac{3}{2}$, and therefore $W(v) = 0$.

We now consider the function $h(x) = \Re(x_1 + i|x_n|)^{\frac{3}{2}}$, which is a $\frac{3}{2}$-homogeneous global solution of the Signorini problem for the Laplacian with zero thin obstacle, and introduce the set

$$H = \{a\Re(\langle x', \nu\rangle + i|x_n|)^{\frac{3}{2}} \mid \nu \in S_1, a \geq 0\}$$

of all multiples and rotations of the function h. The following is the central result of this section.

Theorem 6.3 (Epiperimetric inequality). *There exists $\kappa \in (0,1)$ and $\theta \in (0,1)$ such that if $w \in W^{1,2}(B_1)$ is a homogeneous function of degree $\frac{3}{2}$ such that $w \geq 0$ on B_1' and $\|w-h\|_{W^{1,2}(B_1)} \leq \theta$, then there exists $\zeta \in W^{1,2}(B_1)$ such that $\zeta = w$ on S_1, $\zeta \geq 0$ on B_1' and*

$$W(\zeta) \leq (1-\kappa)W(w).$$

Proof. We argue by contradiction and assume that the result does not hold. Then, there exist sequences of real numbers $\kappa_m \to 0$ and $\theta_m \to 0$, and functions $w_m \in W^{1,2}(B_1)$, homogeneous of degree $\frac{3}{2}$, such that $w_m \geq 0$ on B_1' and

$$\|w_m - h\|_{W^{1,2}(B_1)} \leq \theta_m, \tag{6.1}$$

but such that for every $\zeta \in W^{1,2}(B_1)$ with $\zeta \geq 0$ on B_1', and for which $\zeta = w_m$ on S_1, one has

$$W(\zeta) > (1-\kappa_m)W(w_m). \tag{6.2}$$

With such assumption in place we start by observing that there exists $g_m = a_m\Re(\langle x', \nu_m\rangle + i|x_n|)^{3/2} \in H$ which achieves the minimum distance from w_m to H:

$$\|w_m - g_m\|_{W^{1,2}(B_1)} = \inf_{g\in H} \|w_m - g\|_{W^{1,2}(B_1)}.$$

Indeed, this follows from the simple fact that the set H is locally compact. Combining this inequality with (6.1) we deduce that $\|g_m - h\|_{W^{1,2}(B_1)} \leq 2\theta_m$. As a consequence, we must have that $\nu_m \to e_1$ and $a_m \to 1$. Hence,

$$\left\| \frac{w_m}{a_m} - \Re(\langle x', \nu_m\rangle + i|x_n|)^{\frac{3}{2}} \right\|_{W^{1,2}(B_1)} \leq \frac{\theta_m}{a_m} \to 0.$$

If we rename $\frac{w_m}{a_m} \rightsquigarrow w_m$ and $\frac{\theta_m}{a_m} \rightsquigarrow \theta_m$, and rotate $\mathbb{R}^{n-1}$ to send ν_m to e_1, the renamed functions w_m will be homogeneous of degree $\frac{3}{2}$, nonnegative on B_1', and will satisfy

$$\inf_{g\in H} \|w_m - g\|_{W^{1,2}(B_1)} = \|w_m - h\|_{W^{1,2}(B_1)} \leq \theta_m. \tag{6.3}$$

Moreover, (6.2) will still hold for the renamed w_m, because of the scaling property $W(tw) = t^2 W(w)$ and the invariance of $W(w)$ under rotations in $\mathbb{R}^{n-1}$.

We note explicitly that (6.2) implies in particular that $w_m \neq h$ for every $m \in \mathbb{N}$, as $W(h) = 0$ (see Remark 6.2 above). Thus we may also set

$$\theta_m = \|w_m - h\|_{W^{1,2}(B_1)} > 0 \tag{6.4}$$

for the rest of the proof.

We next want to rewrite (6.2) in a slightly different way, using the properties of function h. Given $\phi \in W^{1,2}(B_1)$, consider the first variation of W at h in the direction of ϕ

$$\delta W(h)(\phi) \overset{\text{def}}{=} \int_{B_1} 2\langle\nabla h, \nabla\phi\rangle - \frac{3}{2}\int_{S_1} 2h\phi, \tag{6.5}$$

where the boundary integrals in (6.5) and thereafter must be interpreted in the sense of traces. To compute $\delta W(h)(\phi)$ we write the first integral in the right-hand side of (6.5) as $\int_{B_1} 2\langle\nabla h,\nabla\phi\rangle = \int_{B_1^+} 2\langle\nabla h,\nabla\phi\rangle + \int_{B_1^-} 2\langle\nabla h,\nabla\phi\rangle$. Now,

$$\begin{aligned}\int_{B_1^+} 2\langle\nabla h,\nabla\phi\rangle &= -\int_{B_1^+} 2(\Delta h)\phi + \int_{S_1^+} 2\phi\langle\nabla h,\nu\rangle + \int_{B_1'} 2\phi\langle\nabla h,\nu_+\rangle\\ &= \int_{S_1^+} 2\phi\langle\nabla h,\nu\rangle + \int_{B_1'} 2\phi\langle\nabla h,\nu_+\rangle,\end{aligned}$$

where we have used the fact that $\Delta h = 0$ in $B_1^\pm$. Since h is homogeneous of degree $\frac{3}{2}$, by Euler's formula we have $\langle\nabla h,\nu\rangle = \frac{3}{2}h$ in $S_1^\pm$. Keeping in mind that on B_1' we have $\nu_\pm = \mp e_n$, we find

$$\int_{B_1^\pm} 2\langle\nabla h,\nabla\phi\rangle = \frac{3}{2}\int_{S_1^\pm} 2\phi h + \int_{B_1'} 2\phi\langle\nabla h,\nu_\pm\rangle = \frac{3}{2}\int_{S_1^\pm} 2\phi h \mp \int_{B_1'} 2\phi\partial_n^\pm h.$$

Since h is even in x_n, so that $\partial_n^- h = -\partial_n^+ h$ in B_1', we conclude

$$\delta W(h)(\phi) = -4\int_{B_1'} \phi\partial_n^+ h. \tag{6.6}$$

If now $\zeta \in W^{1,2}(B_1)$ is a function with $\zeta \geq 0$ on B_1' and such that $\zeta = w_m$ on S_1, by plugging in $\phi = \zeta - h$ into (6.5) and (6.6), we obtain that

$$\begin{aligned}W(\zeta) &= W(\zeta) - W(h) - \delta W(h)(\zeta - h) - 4\int_{B_1'}(\zeta - h)\partial_n^+ h\\ &= \int_{B_1} |\nabla(\zeta - h)|^2 - \frac{3}{2}\int_{S_1}(\zeta - h)^2 - 4\int_{B_1'} \zeta\partial_n^+ h,\end{aligned}$$

where we have used that $W(h) = 0$ and $h\partial_n^+ h = 0$ on B_1'. By using a similar identity for $W(w_m)$, we can rewrite (6.2) as

$$\begin{aligned}(1-\kappa_m)\left[\int_{B_1} |\nabla(w_m - h)|^2 - \frac{3}{2}\int_{S_1}(w_m - h)^2 - 4\int_{B_1'} w_m\partial_n^+ h\right]&\\ < \int_{B_1} |\nabla(\zeta - h)|^2 - \frac{3}{2}\int_{S_1}(\zeta - h)^2 - 4\int_{B_1'} \zeta\partial_n^+ h.&\end{aligned} \tag{6.7}$$

Inequality (6.7) will play a key role in the completion of the proof and will be used repeatedly.

Let us introduce the normalized functions

$$\hat{w}_m = \frac{w_m - h}{\theta_m}.$$

By (6.4) we have

$$\|\hat{w}_m\|_{W^{1,2}(B_1)} = 1 \quad \text{for every } m \in \mathbb{N}.$$

By the weak compactness of the unit sphere in $W^{1,2}(B_1)$ we may assume that

$$\hat{w}_m \to \hat{w} \quad \text{weakly in } W^{1,2}(B_1).$$

Besides, by the compactness of the Sobolev embedding and traces operator from $W^{1,2}(B_1)$ into $L^2(B_1)$, $L^2(B_1')$, $L^2(S_1)$ (see e.g. Theorem 6.3 in [Neč12]), we may assume

$$\hat{w}_m \to \hat{w} \quad \text{strongly in } L^2(B_1),\ L^2(B_1'), \text{ and } L^2(S_1).$$

We then make the following

Claim.

(i) $\hat{w} \equiv 0$;

(ii) $\hat{w}_m \to 0$ *strongly in* $W^{1,2}(B_1)$.

Note that (ii) will give us a contradiction since, by construction $\|\hat{w}_m\|_{W^{1,2}(B_1)} = 1$. Hence, the theorem will follow once we prove the claim.

In what follows we will denote $\Lambda = \Lambda(h)$, the coincidence set of h.

Step 1. We start by showing that there is a constant $C > 0$ such that

$$\Big\| \frac{w_m}{\theta_m^2} \partial_n^+ h \Big\|_{L^1(B_1')} \leq C, \quad \text{for every } m \in \mathbb{N}. \tag{6.8}$$

To this end, we pick a function $\eta \in W_0^{1,\infty}(B_1)$ such that $0 < \eta \leq 1$, and define $\zeta = (1-\eta)w_m + \eta h$. Then, $\zeta = w_m$ on S_1 and $\zeta \geq 0$ on B_1'. Furthermore, $\zeta - h = (1-\eta)(w_m - h)$. We can thus apply (6.7) to such a ζ, obtaining

$$\begin{aligned}
(1-\kappa_m) &\left(\int_{B_1} |\nabla(w_m - h)|^2 - \frac{3}{2} \int_{S_1} (w_m - h)^2 - 4 \int_{B_1'} w_m \partial_n^+ h \right) \\
&< \int_{B_1} |\nabla((1-\eta)(w_m - h))|^2 - \frac{3}{2} \int_{S_1} (1-\eta)^2 (h - w_m)^2 - 4 \int_{B_1'} ((1-\eta) w_m + \eta h) \partial_n^+ h \\
&= \int_{B_1} \big[(1-\eta)^2 |\nabla(w_m - h)|^2 + |\nabla \eta|^2 (w_m - h)^2 - 2(1-\eta)(w_m - h) \langle \nabla \eta, \nabla(w_m - h) \rangle \big] \\
&\qquad\qquad - \frac{3}{2} \int_{S_1} (1-\eta)^2 (h - w_m)^2 - 4 \int_{B_1'} (1-\eta) w_m \partial_n^+ h.
\end{aligned}$$

Dividing by θ_m^2, rearranging terms and using the fact that $\|\hat{w}_m\|_{W^{1,2}(B_1)} = 1$ and that $\partial_n^+ h \leq 0$ on Λ, we obtain

$$\begin{aligned}
4 \int_{B_1'} (\eta - \kappa_m) \frac{w_m}{\theta_m^2} |\partial_n^+ h| &\leq -(1-\kappa_m) \left(\int_{B_1} |\nabla \hat{w}_m|^2 - \frac{3}{2} \int_{S_1} \hat{w}_m^2 \right) \\
&+ \int_{B_1} \big[(1-\eta)^2 |\nabla \hat{w}_m|^2 + |\nabla \eta|^2 \hat{w}_m^2 - 2(1-\eta) \hat{w}_m \langle \nabla \eta, \nabla \hat{w}_m \rangle \big] - \frac{3}{2} \int_{S_1} (1-\eta)^2 \hat{w}_m^2 \leq C,
\end{aligned}$$

where $C > 0$ is independent of $m \in \mathbb{N}$. At this point we choose $\eta(x) = \tilde{\eta}(|x|)$, and let

$$0 < \varepsilon \stackrel{\text{def}}{=} \int_0^1 \tilde{\eta}(r) r^n dr.$$

Since $\kappa_m \to 0$ as $m \to \infty$, possibly passing to a subsequence we can assume that $\kappa_m \leq \frac{\varepsilon}{2}(n+1)$ for every $m \in \mathbb{N}$. With such choice we have

$$\int_0^1 (\tilde{\eta}(r) - \kappa_m) r^n dr \geq \frac{\varepsilon}{2}, \quad m \in \mathbb{N}.$$

Using the fact that w_m and h are homogeneous of degree $\frac{3}{2}$, we thus obtain

$$C \geq 4 \int_{B_1'} (\eta - \kappa_m) \frac{w_m}{\theta_m^2} |\partial_n^+ h| = 4 \left(\int_0^1 (\tilde{\eta}(r) - \kappa_m) r^n dr \right) \int_{S_1'} \frac{w_m}{\theta_m^2} |\partial_n^+ h| \geq 2\varepsilon \int_{S_1'} \frac{w_m}{\theta_m^2} |\partial_n^+ h|,$$

which, again by the homogeneity of w_m and h, and the fact that $w_m \geq 0$ on B_1', proves (6.8).

Step 2. We next show that

$$\Delta \hat{w} = 0, \quad \text{in } B_1 \setminus \Lambda. \tag{6.9}$$

To establish (6.9) it will suffice to show that for any ball B, such that its concentric double $2B \Subset B_1 \setminus \Lambda$, and for any function $\phi \in W^{1,2}(B)$ such that $\phi - \hat{w} \in W_0^{1,2}(B)$, one has

$$\int_B |\nabla \hat{w}|^2 \leq \int_B |\nabla \phi|^2.$$

To begin, we fix a function $\phi \in L^\infty(B_1) \cap W^{1,2}(B)$, and we consider

$$\zeta = \eta(h + \theta_m \phi) + (1-\eta) w_m,$$

where $\eta \in C_0^\infty(B_1 \setminus \Lambda)$, $0 \leq \eta \leq 1$. Notice that on S_1, $\zeta = w_m$, and, since $\phi \in L^\infty(B_1)$ and $\eta \in C_0^\infty(B_1 \setminus \Lambda)$, for m large enough we have $\zeta \geq 0$ on B_1'. For such sufficiently large m's, we can thus use the function ζ in (6.7), obtaining

$$\begin{aligned}
(1-\kappa_m) &\left(\int_{B_1} |\nabla(w_m - h)|^2 - \frac{3}{2}\int_{S_1} (w_m - h)^2 - 4\int_{B_1'} w_m \partial_n^+ h \right) \\
&< \int_{B_1} |\nabla((1-\eta)(w_m - h) + \eta\theta_m\phi)|^2 - \frac{3}{2}\int_{S_1} ((1-\eta)(w_m-h) + \eta\theta_m\phi)^2 \\
&\qquad - 4\int_{B_1'} (\eta(h + \theta_m\phi) + (1-\eta)w_m)\partial_n^+ h.
\end{aligned}$$

Dividing by θ_m^2 and recalling that $h\partial_n^+ h = 0$ in B_1', we obtain

$$\begin{aligned}
(1-\kappa_m) &\left(\int_{B_1} |\nabla \hat{w}_m|^2 - \frac{3}{2}\int_{S_1} \hat{w}_m^2 - 4\int_{B_1'} \frac{w_m}{\theta_m^2} \partial_n^+ h \right) \\
&< \int_{B_1} \left[|\nabla(\eta\phi)|^2 + |\nabla((1-\eta)\hat{w}_m)|^2 + 2\langle \nabla(\eta\phi), \nabla((1-\eta)\hat{w}_m)\rangle \right] \\
&\qquad - 4\int_{B_1'} \left[\frac{\eta\phi}{\theta_m} \partial_n^+ h + (1-\eta)\frac{w_m}{\theta_m^2}\partial_n^+ h \right] - \frac{3}{2}\int_{S_1} ((1-\eta)\hat{w}_m + \eta\phi)^2 \\
&= \int_{B_1} \left[|\nabla(\eta\phi)|^2 + |\nabla((1-\eta)\hat{w}_m)|^2 + 2\langle \nabla(\eta\phi), \nabla((1-\eta)\hat{w}_m)\rangle \right] - 4\int_{B_1'} \frac{w_m}{\theta_m^2}\partial_n^+ h - \frac{3}{2}\int_{S_1} \hat{w}_m^2,
\end{aligned}$$

since $\eta \in C_0^\infty(B_1 \setminus \Lambda)$. Hence

$$\begin{aligned}\int_{B_1} |\nabla \hat{w}_m|^2 < \kappa_m \int_{B_1} |\nabla \hat{w}_m|^2 + \frac{3}{2}(1-\kappa_m) \int_{S_1} \hat{w}_m^2 + 4(1-\kappa_m) \int_{B_1'} \frac{w_m}{\theta_m^2} \partial_n^+ h \\ + \int_{B_1} \left[|\nabla(\eta\phi)|^2 + |\nabla((1-\eta)\hat{w}_m)|^2 + 2\langle \nabla(\eta\phi), \nabla((1-\eta)\hat{w}_m)\rangle \right] \\ - 4 \int_{B_1'} \frac{w_m}{\theta_m^2} \partial_n^+ h - \frac{3}{2} \int_{S_1} \hat{w}_m^2 \\ = \kappa_m \int_{B_1} |\nabla \hat{w}_m|^2 - \frac{3}{2}\kappa_m \int_{S_1} \hat{w}_m^2 - 4 \int_{B_1'} \kappa_m \frac{w_m}{\theta_m^2} \partial_n^+ h \\ + \int_{B_1} \left[|\nabla(\eta\phi)|^2 + |\nabla((1-\eta)\hat{w}_m)|^2 + 2\langle \nabla(\eta\phi), \nabla((1-\eta)\hat{w}_m)\rangle \right] \\ \leq C\kappa_m + \int_{B_1} \left[|\nabla(\eta\phi)|^2 + |\nabla((1-\eta)\hat{w}_m)|^2 + 2\langle \nabla(\eta\phi), \nabla((1-\eta)\hat{w}_m)\rangle \right].\end{aligned}$$

Therefore

$$\begin{aligned}\int_{B_1} (1-(1-\eta)^2)|\nabla \hat{w}_m|^2 \leq C\kappa_m + \int_{B_1} \Big[|\nabla(\eta\phi)|^2 + \hat{w}_m^2 |\nabla \eta|^2 \\ - 2(1-\eta)\hat{w}_m \langle \nabla\eta, \nabla \hat{w}_m \rangle + 2\langle \nabla(\eta\phi), \nabla((1-\eta)\hat{w}_m)\rangle \Big].\end{aligned}$$

Passing to the limit $m \to \infty$ we obtain

$$\begin{aligned}\int_{B_1} (1-(1-\eta)^2)|\nabla \hat{w}|^2 \leq \int_{B_1} \Big[|\nabla(\eta\phi)|^2 + \hat{w}^2 |\nabla \eta|^2 \\ - 2(1-\eta)\hat{w} \langle \nabla\eta, \nabla \hat{w} \rangle + 2\langle \nabla(\eta\phi), \nabla((1-\eta)\hat{w})\rangle \Big].\end{aligned} \tag{6.10}$$

Notice that

$$\begin{aligned}\int_{B_1} |\nabla(\eta\phi + (1-\eta)\hat{w})|^2 = \int_{B_1} \left[|\nabla(\eta\phi)|^2 + |\nabla((1-\eta)\hat{w})|^2 + 2\langle \nabla(\eta\phi), \nabla((1-\eta)\hat{w}\rangle \right] \\ = \int_{B_1} \Big[|\nabla(\eta\phi)|^2 + \hat{w}^2|\nabla\eta|^2 + (1-\eta)^2|\nabla\hat{w}|^2 - 2\hat{w}(1-\eta)\langle \nabla\hat{w}, \nabla\eta\rangle \\ + 2\langle \nabla(\eta\phi), \nabla((1-\eta)\hat{w})\rangle \Big],\end{aligned}$$

hence (6.10) gives us,

$$\int_{B_1} |\nabla \hat{w}|^2 \leq \int_{B_1} |\nabla(\eta\phi + (1-\eta)\hat{w})|^2.$$

By approximation we can drop the condition $\phi \in L^\infty(B_1)$ and by considering open balls $B \Subset B_1 \setminus \Lambda$ we may choose $\eta = 1$ in B and $\phi = \hat{w}$ outside B. This will give

$$\int_{B_1} |\nabla \hat{w}|^2 \leq \int_{B} |\nabla \phi|^2 + \int_{B_1 \setminus B} |\nabla \hat{w}|^2,$$

hence

$$\int_{B} |\nabla \hat{w}|^2 \leq \int_{B} |\nabla \phi|^2,$$

which proves the harmonicity of $\hat{w}$ in B.

Step 3. We next want to prove that

$$\hat{w} = 0 \quad \mathcal{H}^{n-1}\text{-a.e. in } \Lambda. \tag{6.11}$$

We note that for the function h we have $\partial_n^+ h(x',0) \neq 0$ for every $(x',0) \in \Lambda^\circ$ (interior of Λ in $\mathbb{R}^{n-1}$). Therefore, given $\omega \Subset \Lambda^\circ$, there exists a constant $C_\omega > 0$ such that $|\partial_n^+ h(x',0)| \geq C_\omega$ for every $(x',0) \in \omega$. At points $(x',0) \in \Lambda^\circ$, we can thus write

$$\hat{w}_m = \frac{w_m - h}{\theta_m} = \frac{w_m}{\theta_m^2} \partial_n^+ h \frac{\theta_m}{\partial_n^+ h}.$$

This gives

$$\int_\omega |\hat{w}_m| \leq \frac{\theta_m}{C_\omega} \int_\omega \frac{w_m}{\theta_m^2} |\partial_n^+ h| \leq \frac{C\theta_m}{C_\omega},$$

where in the last inequality we have used (6.8) in Step 1 above. Since $\theta_m \to 0$, we conclude that $\|\hat{w}_m\|_{L^1(\omega)} \to 0$ as $m \to \infty$. By the arbitrariness of $\omega \Subset \Lambda^\circ$ we infer that, in particular, we must have

$$\hat{w}_m(x',0) \to 0, \quad \mathcal{H}^{n-1}\text{-a.e. } (x',0) \in \Lambda, \tag{6.12}$$

which proves (6.11).

Step 4 (Proof of (i)). We next show that

$$\hat{w}_m \to 0, \quad \text{weakly in } W^{1,2}(B_1), \tag{6.13}$$

or, equivalently, $\hat{w} = 0$. We begin by observing that, since the $\hat{w}_m$'s are homogeneous of degree $\frac{3}{2}$, their weak limit $\hat{w}$ is also homogeneous of degree $\frac{3}{2}$. Combining this observation with Steps 2 and 3 above, we then have the following properties for $\hat{w}$:

(i) $\Delta \hat{w} = 0$ in $B_1 \setminus \Lambda$;
(ii) $\hat{w} = 0$ $\mathcal{H}^{n-1}$-a.e. on Λ;
(iii) $\hat{w}$ is homogeneous of degree $\frac{3}{2}$.

We next obtain an explicit representation for $\hat{w}$. First, we note that $\hat{w}$ is Hölder continuous up to the coincidence set Λ of h. Indeed, this can be seen by making a bi-Lipschitz transformation $T : B_1 \setminus \Lambda \to B_1^+$ as in (b) on p. 501 of [AC85]. The function $\tilde{w} = \hat{w} \circ T^{-1} \in W^{1,2}(B_1^+)$ solves a uniformly elliptic equation in divergence form

$$\operatorname{div}(b(x)\nabla \tilde{w}) = 0 \quad \text{in } B_1^+$$

with bounded measurable coefficients $b(x) = [b_{ij}(x)]$. We will also have

$$\tilde{w} = 0 \quad \text{on } B_1'$$

in the sense of traces. Then, by the boundary version of De Giorgi-Nash-Moser regularity theorem (see, e.g., Theorem 8.29 in [GT01]) we have that $\tilde{w}$ is C^γ up to B_1' for some $\gamma > 0$. Since $\hat{w}(x) = \tilde{w}(T(x))$, this implies that $\hat{w} \in C^\gamma(B_1)$.

Once we know that $\hat{w} \in C^\gamma(B_1)$, together with (i)–(iii) above, we can apply a theorem of De Silva and Savin on an expansion of harmonic functions in slit domains, see Theorem 3.3 in [DSS14a] (and also Theorem 4.5 in the same paper) which implies that there are constants $a_1, a_2, \ldots, a_{n-1}$, b, and c such that, for some $\alpha \in (0,1)$,

$$|\hat{w}(x) - P_0(x)U_0(x) - cx_n| = O(|x|^{3/2+\alpha}),$$

where

$$U_0(x) = \frac{1}{\sqrt{2}} \sqrt{x_1 + \sqrt{x_1^2 + x_n^2}} = \Re(x_1 + i|x_n|)^{1/2},$$

and

$$P_0(x) = \sum_{k=1}^{n-1} a_k x_k + b\sqrt{x_1^2 + x_n^2}.$$

Since $\hat{w}$ is 3/2-homogeneous, we must have $c = 0$ and thus

$$\hat{w}(x) = P_0(x)U_0(x).$$

Now, a direct computation shows that such $\hat{w}$ will be harmonic in $B_1 \setminus \Lambda$ only if

$$a_1 + 2b = 0,$$

which implies the representation

$$\begin{aligned}\hat{w}(x) &= \frac{a_1}{2}\Re(x_1 + i|x_n|)^{3/2} + \sum_{j=2}^{n-1} a_j x_j \Re(x_1 + i|x_n|)^{1/2}, \\ &= \frac{a_1}{2}h(x) + \sum_{j=2}^{n-1} a_j x_j U_0(x).\end{aligned}$$

We next show that all constants $a_j = 0$, $j = 1, \ldots, n-1$. To simplify the notation we will write $\|\cdot\| = \|\cdot\|_{W^{1,2}(B_1)}$. We then use the fact that

$$\|w_m - g\|^2 \geq \|w_m - h\|^2 \quad \text{for all} \quad g \in H.$$

Recalling that $\hat{w}_m = \frac{w_m - h}{\theta_m}$, we can write this as

$$\|\theta_m \hat{w}_m + h - g\|^2 \geq \|\theta_m \hat{w}_m\|^2,$$

or

$$2\theta_m \langle \hat{w}_m, h - g\rangle + \|h - g\|^2 \geq 0.$$

Therefore,

$$\langle \hat{w}_m, g - h\rangle \leq \frac{\|h - g\|^2}{2\theta_m}. \tag{6.14}$$

Applying this to $g = (1 + \theta_m^2)h$, we obtain

$$\langle \hat{w}_m, h\rangle \leq \frac{\theta_m}{2}\|h\|^2.$$

Letting $m \to \infty$ we arrive at

$$\langle \hat{w}, h\rangle = \frac{a_1}{2}\|h\|^2 \leq 0.$$

This implies that $a_1 \leq 0$. Using the same argument for $g = (1 - \theta_m^2)h$ allows us to conclude that also $-a_1 \leq 0$, and therefore $a_1 = 0$. Further, rewriting (6.14) as

$$\left\langle \hat{w}_m, \frac{g - h}{\theta_m^2} \right\rangle \leq \frac{\theta_m}{2} \left\| \frac{g - h}{\theta_m^2} \right\|^2,$$

and taking for $j = 2, \ldots, n-1$

$$g = \Re(x_1 \cos(\theta_m^2) + \sin(\theta_m^2)x_j + i|x_n|)^{3/2},$$

in such inequality, by letting $m \to \infty$ we obtain that

$$\frac{3}{2}\langle \hat{w}, x_j U_0\rangle = \frac{3}{2}a_j \|x_j U_0\|^2 \leq 0.$$

(We note here that $\langle x_i U_0, x_j U_0\rangle = 0$ for $i, j = 2, \ldots, n-1$, $i \neq j$ and that

$$\frac{\Re(x_1\cos(\theta)+\sin(\theta)x_j+i|x_n|)^{3/2}-\Re(x_1+i|x_n|)^{3/2}}{\theta}\to\frac{3}{2}x_jU_0(x)$$

as $\theta\to 0$, strongly in $W^{1,2}(B_1)$.) Hence $a_j\leq 0$. Replacing x_j with $-x_j$ in the above argument, we also obtain $-a_j\leq 0$. Thus, $a_j=0$ for all $j=1,\dots,n-1$, which implies $\hat w=0$ and completes the proof of (6.13).

Step 5 (Proof of (ii)): Finally, we claim that, on a subsequence,

$$\hat w_m\to 0\quad\text{strongly in } W^{1,2}(B_1). \tag{6.15}$$

Since we already have the strong convergence $\hat w_m\to\hat w=0$ in $L^2(B_1)$, we are left with proving

$$\nabla\hat w_m\to 0\quad\text{strongly in } L^2(B_1). \tag{6.16}$$

To this end, we pick $\eta\in C_0^{0,1}(B_1)$, $0\leq\eta\leq 1$, and consider $\zeta=(1-\eta)w_m+\eta h$. Clearly, $\zeta=w_m$ on S_1, $\zeta\geq 0$ on B_1', and $\zeta-h=(1-\eta)(w_m-h)$. Applying (6.7) with this choice of ζ we obtain

$$\begin{aligned}(1-\kappa_m)\left[\int_{B_1}|\nabla(w_m-h)|^2-\frac{3}{2}\int_{S_1}(w_m-h)^2-4\int_{B_1'}w_m\partial_n^+h\right]\\ <\int_{B_1}|(1-\eta)\nabla(w_m-h)-\nabla\eta(w_m-h)|^2\\ -\frac{3}{2}\int_{S_1}(1-\eta)^2(w_m-h)^2-4\int_{B_1'}(1-\eta)w_m\partial_n^+h.\end{aligned}$$

Dividing by θ_m^2, and recalling that $\hat w_m=\frac{w_m-h}{\theta_m}$, we obtain

$$\begin{aligned}(1-\kappa_m)\left(\int_{B_1}|\nabla\hat w_m|^2-\frac{3}{2}\int_{S_1}\hat w_m^2-4\int_{B_1'}\frac{w_m}{\theta_m^2}\partial_n^+h\right)\\ <\int_{B_1}\left[(1-\eta)^2|\nabla\hat w_m|^2+\hat w_m^2|\nabla\eta|^2-2(1-\eta)\hat w_m\langle\nabla\hat w_m,\nabla\eta\rangle\right]\\ -\frac{3}{2}\int_{S_1}(1-\eta)^2\hat w_m^2-4\int_{B_1'}(1-\eta)\frac{w_m}{\theta_m^2}\partial_n^+h.\end{aligned}$$

This gives

$$
\begin{aligned}
\int_{B_1} |\nabla \hat w_m|^2 - 4\int_{B_1'} \frac{w_m}{\theta_m^2}\partial_n^+ h \qquad&\\
\le \int_{B_1} &\left[(1-\eta)^2|\nabla \hat w_m|^2 + |\nabla\eta|^2 \hat w_m^2 - 2(1-\eta)\hat w_m\langle\nabla \hat w_m,\nabla\eta\rangle\right]\\
&-\frac{3}{2}\int_{S_1}(1-\eta)^2\hat w_m^2 - 4\int_{B_1'}(1-\eta)\frac{w_m}{\theta_m^2}\partial_n^+ h + (1-\kappa_m)\frac{3}{2}\int_{S_1}\hat w_m^2\\
&\qquad + \kappa_m\left(\int_{B_1}|\nabla \hat w_m|^2 - 4\int_{B_1'}\frac{w_m}{\theta_m^2}\partial_n^+ h\right)\\
= \int_{B_1} &\left[(1-\eta)^2|\nabla \hat w_m|^2 + |\nabla\eta|^2 \hat w_m^2 - 2(1-\eta)\hat w_m\langle\nabla \hat w_m,\nabla\eta\rangle\right]\\
&+\kappa_m\left(\int_{B_1}|\nabla \hat w_m|^2 - 4\int_{B_1'}\frac{w_m}{\theta_m^2}\partial_n^+ h - \frac{3}{2}\int_{S_1}\hat w_m^2\right)\\
&\qquad + \frac{3}{2}\left(1-(1-\eta)^2\right)\int_{S_1}\hat w_m^2 - 4\int_{B_1'}(1-\eta)\frac{w_m}{\theta_m^2}\partial_n^+ h.
\end{aligned}
$$

If in this inequality we use the fact that $\|\nabla \hat w_m\|_{L^2(B_1)} \le \|\hat w_m\|_{W^{1,2}(B_1)} = 1$, and that $\frac{w_m}{\theta_m^2}\partial_n^+ h$ is uniformly bounded in $L^1(B_1')$, a fact which we have proved in (6.8) of Step 1, we obtain

$$
\begin{aligned}
\int_{B_1} |\nabla \hat w_m|^2 - 4\int_{B_1'} \frac{w_m}{\theta_m^2}\partial_n^+ h \qquad&\\
\le \int_{B_1} &(1-\eta)^2|\nabla \hat w_m|^2 + |\nabla\eta|^2 \hat w_m^2 - 2(1-\eta)\hat w_m\langle\nabla \hat w_m,\nabla\eta\rangle\\
&\qquad + C\kappa_m + \frac{3}{2}\int_{S_1}\hat w_m^2 - 4\int_{B_1'}(1-\eta)\frac{w_m}{\theta_m^2}\partial_n^+ h.
\end{aligned}
\tag{6.17}
$$

We now make the choice in (6.17) of

$$
\eta(x) = \begin{cases} 1, & \text{if } |x| \le \frac{1}{2},\\ 2(1-|x|), & \text{if } \frac{1}{2} < |x| < 1,\\ 0, & \text{if } |x| \ge 1,\end{cases}
$$

we obtain

$$
\begin{aligned}
\int_{B_{\frac12}} |\nabla \hat w_m|^2 &\le \int_{B_1}\left[|\nabla\eta|^2\hat w_m^2 - 2(1-\eta)\hat w_m\langle\nabla \hat w_m,\nabla\eta\rangle\right] + \frac{3}{2}\int_{S_1}\hat w_m^2 + C\kappa_m + 4\int_{B_1'}\eta\frac{w_m}{\theta_m^2}\partial_n^+ h\\
&\le \int_{B_1}\left[|\nabla\eta|^2\hat w_m^2 - 2(1-\eta)\hat w_m\langle\nabla \hat w_m,\nabla\eta\rangle\right] + \frac{3}{2}\int_{S_1}\hat w_m^2 + C\kappa_m,
\end{aligned}
$$

since $\eta, w_m \ge 0$ and $\partial_n^+ h \le 0$. We thus conclude that

$$
\int_{B_{\frac12}} |\nabla \hat w_m|^2 \le \int_{B_1}\left[|\nabla\eta|^2\hat w_m^2 - 2(1-\eta)\hat w_m\langle\nabla \hat w_m,\nabla\eta\rangle\right] + \frac{3}{2}\int_{S_1}\hat w_m^2 + C\kappa_m. \tag{6.18}
$$

We now observe that, since $\hat{w}_m$ is homogeneous of degree $3/2$, and thus $\nabla\hat{w}_m$ is homogeneous of degree $1/2$, we have

$$\int_{B_1} |\nabla\hat{w}_m|^2 = 2^{n+1}\int_{B_{\frac{1}{2}}} |\nabla\hat{w}_m|^2.$$

Using this identity in (6.18) we conclude that

$$\int_{B_1} |\nabla\hat{w}_m|^2 \le 2^{n+1}\left(\int_{B_1}\left[|\nabla\eta|^2\hat{w}_m^2 - 2(1-\eta)\hat{w}_m\langle\nabla\hat{w}_m,\nabla\eta\rangle\right] + \frac{3}{2}\int_{S_1}\hat{w}_m^2 + C\kappa_m\right).$$

To complete the proof of (6.15), and consequently of Theorem 6.3, all we need to do at this point is to observe that, on a subsequence, the right-hand side of the latter inequality converges to 0 as $m\to\infty$. This follows from the facts that $\kappa_m\to 0$, $\|\hat{w}_m\|_{L^2(B_1)}\to 0$, $\|\hat{w}_m\|_{L^2(S_1)}\to 0$, and $\|\nabla\hat{w}_m\|_{L^2(B_1)}\le 1$.

This completes the proof of the claim and that of the theorem. □

7. $C^{1,\beta}$ regularity of the regular part of the free boundary

In this final section we combine Theorems 4.3 and 6.3 to establish the $C^{1,\beta}$ regularity of the regular part of the free boundary. We will consider two types of scalings: the Almgren one, defined in (3.2), and the homogeneous scalings defined in (5.1), which are suited for the study of regular free boundary points, i.e.,

$$v_r(x) = \frac{v(rx)}{r^{3/2}}. \tag{7.1}$$

Throughout this section we continue to use the notation $h(x) = \Re(x_1 + i|x_n|)^{\frac{3}{2}}$ adopted in Section 6. The symbol $\theta > 0$ will be used to exclusively denote the constant in the epiperimetric inequality of Theorem 6.3 above.

In Lemma 4.2 above we showed that our Weiss type functional $W_L(v,r)$ is bounded, when v is the solution to the problem (1.12)–(1.15). In the course of the proof of the next lemma we establish the much more precise statement that $W_L(v,r)\le Cr^\gamma$, for appropriate constants $C,\gamma>0$. This gain is possible because of the assumption, in Lemma 7.1 below, that the scalings v_r have the epiperimetric property, i.e., the conclusion of the epiperimetric inequality holds for their extensions as $3/2$-homogeneous functions in B_1.

Lemma 7.1. *Let v be the solution of the thin obstacle problem* (1.12)–(1.15)*, and suppose that $0\in\Gamma_{3/2}(v)$. Assume the existence of radii $0\le s_0<r_0<1$ such that for every $s_0\le r\le r_0$, if we extend $v_r\big|_{S_1}$ as a $3/2$-homogeneous function in B_1, call it w_r, then there exists a function $\zeta_r\in W^{1,2}(B_1)$ such that $\zeta_r\ge 0$ in B_1', $\zeta_r = v_r$ on S_1 and*

$$W(\zeta_r)\le(1-\kappa)W(w_r),$$

where κ is the constant in the epiperimetric inequality. Then, there exist universal constants $C,\gamma>0$ such that for every $s_0\le s\le t\le r_0$ one has

$$\int_{S_1}|v_t - v_s|\le Ct^\gamma. \tag{7.2}$$

Proof. As before, we let $L = \operatorname{div}(A\nabla\cdot)$. The main idea of the proof of (7.2) is to relate $\int_{S_1}|v_t-v_s|$ with the Weiss type functional $W_L(v,r)$ defined in (4.1) above, and then control the latter in the following way:

$$W_L(v,t)\le Ct^\gamma,\quad 0<t<r_0. \tag{7.3}$$

More specifically, combining equations (7.11) and (7.12) proved below, we obtain the following

$$\int_{S_1} |v_t - v_s| \le C(n) \int_s^t r^{-1/2} \left(\frac{d}{dr} W_L(v,r) + C r^{-1/2} \right)^{1/2} dr. \tag{7.4}$$

After using Hölder's inequality in the right-hand side of (7.4) we obtain

$$\begin{aligned}
\int_{S_1} |v_t - v_s| &\le C \left(\int_s^t r^{-1} dr \right)^{1/2} \left(\int_s^t \frac{d}{dr} W_L(v,r) + C r^{-1/2} dr \right)^{1/2} \\
&\le C \left(\log \frac{t}{s} \right)^{1/2} \left(W_L(v,t) - W_L(v,s) + C(t^{1/2} - s^{1/2}) \right)^{1/2} \\
&\le C \left(\log \frac{t}{s} \right)^{1/2} \left(C t^{\gamma} + C s^{1/2} + C(t^{1/2} - s^{1/2}) \right)^{1/2} \\
&\le C \left(\log \frac{t}{s} \right)^{1/2} t^{\frac{\gamma}{2}},
\end{aligned}$$

where we have used (7.3), and we have estimated $-W_L(v,s) \le C s^{1/2}$ using Corollary 4.5. With this estimate in hands we now use a dyadic argument. Assume that $s \in [2^{-\ell}, 2^{-\ell+1})$, $t \in [2^{-h}, 2^{-h+1})$ with $h \le \ell$ and apply the estimate above iteratively. We obtain

$$\begin{aligned}
\int_{S_1} |v_s - v_t| &\le \int_{S_1} |v_s - v_{2^{-\ell+1}}| + \cdots + \int_{S_1} |v_{2^{-h}} - v_t| \\
&\le C \log^{1/2} \left(\frac{2^{-\ell+1}}{s} \right) \left(2^{-\ell+1} \right)^{\frac{\gamma}{2}} + \cdots + C \log^{1/2} \left(\frac{t}{2^{-h}} \right) t^{\frac{\gamma}{2}} \\
&\le C (\log 2)^{1/2} \left(2^{\frac{\gamma}{2}} \right)^{-\ell+1} + \cdots + C (\log 2)^{1/2} t^{\frac{\gamma}{2}} \\
&\le C (\log 2)^{1/2} \sum_{j=h}^{\ell-1} \left(2^{\frac{\gamma}{2}} \right)^{-j} + C (\log 2)^{1/2} t^{\frac{\gamma}{2}} \\
&\le C (\log 2)^{1/2} \left(2^{\frac{\gamma}{2}} \right)^{-h} + C (\log 2)^{1/2} t^{\frac{\gamma}{2}} \le C t^{\frac{\gamma}{2}},
\end{aligned}$$

which yields the sought for conclusion (7.2).

In order to complete the proof of the lemma we are thus left with proving (7.3) and (7.4). Our first step will be to prove (7.3) since the computations leading to such estimate also give (7.4), as we will show below. We will establish (7.3) by proving (see (7.10) below) the following estimate

$$\frac{d}{dr} W_L(v,r) \ge \frac{n+1}{r} \frac{\kappa}{1-\kappa} W_L(v,r) - C r^{-1/2},$$

where κ is the constant in the epiperimetric inequality. With this objective in mind, we recall that

$$W_L(v,r) = \frac{I_L(v,r)}{r^{n+1}} - \frac{3}{2} \frac{H_L(v,r)}{r^{n+2}},$$

see (4.2). To simplify the notation we write $I = I_L$ and $H = H_L$. We start by observing that combining Lemma 2.1 with the observation that $L|x| = \operatorname{div}(A(x)\nabla r) = \frac{n-1}{|x|}(1 + O(|x|))$ (see Lemma 4.1 in [GSVG14]), we obtain the following estimate for $H'(r)$:

$$H'(r) - \left(\frac{n-1}{r} + O(1) \right) H(r) = 2I(r). \tag{7.5}$$

In the computations that follow we will estimate $\frac{d}{dr}W_L(v,r)$ using formula (7.5), estimates (2.16), (2.17), as well as the identity $I(r) = D(r) + \int_{B_r} vf$, which gives $I'(r) = \int_{S_r}\langle A\nabla v, \nabla v\rangle + \int_{S_r} vf$. We thus have

$$\begin{aligned}
\frac{d}{dr}W_L(v,r) &\overset{(4.2)}{=} \frac{I'(r)}{r^{n+1}} - \frac{n+1}{r^{n+2}}I(r) - \frac{3}{2r^{n+2}}H'(r) + \frac{3(n+2)}{2r^{n+3}}H(r)\\
&\overset{(7.5)}{\geq} \frac{1}{r^{n+1}}\int_{S_r}\langle A\nabla v, \nabla v\rangle - \frac{n+1}{r^{n+2}}D(r) + \frac{3(n+2)}{2r^{n+3}}H(r)\\
&\quad - \frac{3}{2r^{n+2}}\left(\frac{n-1}{r}H(r) + 2\int_{S_r} v\langle A\nu, \nabla v\rangle + CH(r)\right)\\
&\quad + \frac{1}{r^{n+1}}\int_{S_r} vf - \frac{n+1}{r^{n+2}}\int_{B_r} vf\\
&\overset{(2.16)}{\geq} \frac{1}{r^{n+1}}\int_{S_r}\langle A\nabla v, \nabla v\rangle - \frac{n+1}{r^{n+2}}D(r) + \frac{9}{2r^{n+3}}H(r) - \frac{3}{r^{n+2}}\int_{S_r} v\langle A\nu, \nabla v\rangle - Cr^{-1/2}\\
&= \frac{1}{r^{n+1}}\int_{S_r}\langle A\nabla v, \nabla v\rangle - \frac{n+1}{r}W_L(v,r) - \frac{3(n-2)}{2r^{n+3}}H(r) - \frac{3}{r^{n+2}}\int_{S_r} v\langle A\nu, \nabla v\rangle - Cr^{-1/2},
\end{aligned}$$

where using (2.16) we have estimated $CH(r) \leq Cr^{n+2}$, $|\int_{S_r} vf| \leq Cr^{n+\frac{1}{2}}$. Now

$$\begin{aligned}
H(r) &= \int_{S_r}\mu v^2 = \int_{S_r} v^2 + \int_{S_r}(\mu - 1)v^2\\
&\leq \int_{S_r} v^2 + r\int_{S_r} v^2 \overset{(2.16)}{\leq} \int_{S_r} v^2 + Crr^{n-1+3} = \int_{S_r} v^2 + Cr^{n+3}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int_{S_r}\langle A\nabla v, \nabla v\rangle &= \int_{S_r}|\nabla v|^2 + \int_{S_r}\langle (A(x) - A(0))\nabla v, \nabla v\rangle\\
&\leq \int_{S_r}|\nabla v|^2 + r\int_{S_r}|\nabla v|^2 \overset{(2.16)}{\leq} \int_{S_r}|\nabla v|^2 + Cr^{n+1}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\int_{S_r} v\langle A\nu, \nabla v\rangle &= \int_{S_r} v\langle \nu, \nabla v\rangle + \int_{S_r} v\langle (A(x) - A(0))\nu, \nabla v\rangle\\
&\leq \int_{S_r} v\langle \nu, \nabla v\rangle + Cr\int_{S_r}|v||\nabla v|\\
&\leq \int_{S_r} v\langle \nu, \nabla v\rangle + Cr\left(\int_{S_r} v^2\int_{S_r}|\nabla v|^2\right)^{\frac{1}{2}}\\
&\overset{(2.16)}{\leq} \int_{S_r} v\langle \nu, \nabla v\rangle + Crr^{\frac{n-1+3}{2}}r^{\frac{n-1+1}{2}}\\
&= \int_{S_r} v\langle \nu, \nabla v\rangle + Cr^{n+2}.
\end{aligned}$$

This implies

$$\begin{aligned}\frac{d}{dr}W_L(v,r) &\geq \frac{1}{r^{n+1}}\int_{S_r}|\nabla v|^2 - \frac{n+1}{r}W_L(v,r) - \frac{3(n-2)}{2r^{n+3}}\int_{S_r}v^2 - \frac{3}{r^{n+2}}\int_{S_r}v\langle\nu,\nabla v\rangle - Cr^{-1/2}\\ &= -\frac{n+1}{r}W_L(v,r) + \frac{1}{r}\int_{S_1}|\nabla v_r|^2 - \frac{3(n-2)}{2r}\int_{S_1}v_r^2 - \frac{3}{r}\int_{S_1}v_r\langle\nu,\nabla v_r\rangle - Cr^{-1/2}\\ &= -\frac{n+1}{r}W_L(v,r) + \frac{1}{r}\int_{S_1}\left((\langle\nu,\nabla v_r\rangle - \frac{3}{2}v_r)^2 + |\partial_\tau v_r|^2 - \frac{3}{2}\left(n-\frac{1}{2}\right)v_r^2\right) - Cr^{-1/2},\end{aligned}$$

where $\partial_\tau v_r$ is the tangential derivative of v_r along S_1. Let w_r be the $\frac{3}{2}$-homogeneous extension of $v_r\big|_{S_1}$, then

$$\int_{S_1}\left(|\partial_\tau v_r|^2 - \frac{3}{2}\left(n-\frac{1}{2}\right)v_r^2\right) = \int_{S_1}\left(|\partial_\tau w_r|^2 - \frac{3}{2}\left(n-\frac{1}{2}\right)w_r^2\right).$$

Recalling that w_r is homogeneous of degree $3/2$, we have on S_1 that $\langle\nabla w_r,\nu\rangle = \frac{3}{2}w_r$. This gives

$$\begin{aligned}\int_{S_1}\left(|\partial_\tau w_r|^2 - \frac{3}{2}\left(n-\frac{1}{2}\right)w_r^2\right) &= \int_{S_1}\left(|\nabla w_r|^2 - \langle\nabla w_r,\nu\rangle^2 - \frac{3}{2}\left(n-\frac{1}{2}\right)w_r^2\right)\\ &= \int_{S_1}\left(|\nabla w_r|^2 - \frac{9}{4}w_r^2 - \frac{3}{2}\left(n-\frac{1}{2}\right)w_r^2\right) = \int_{S_1}\left(|\nabla w_r|^2 - \frac{3}{2}(n+1)\,w_r^2\right).\end{aligned}$$

Now, since again by the homogeneity of w_r,

$$\int_{S_1}|\nabla w_r|^2 = (n+1)\int_{B_1}|\nabla w_r|^2,$$

we conclude that

$$\int_{S_1}\left(|\partial_\tau w_r|^2 - \frac{3}{2}\left(n-\frac{1}{2}\right)w_r^2\right) = (n+1)W(w_r,1),$$

where we recall that, by definition (4.1) above,

$$W(w,s) = W_\Delta(w,s) = \frac{1}{s^{n+1}}\int_{B_s}|\nabla w|^2 - \frac{3}{2s^{n+2}}\int_{S_s}w^2.$$

Hence

$$\frac{d}{dr}W_L(v,r) \geq \frac{n+1}{r}\left(W(w_r,1) - W_L(v,r)\right) + \frac{1}{r}\int_{S_1}(\langle\nu,\nabla v_r\rangle - \frac{3}{2}v_r)^2 - Cr^{-1/2}. \tag{7.6}$$

By the hypothesis, for every $s_0 \leq r \leq r_0$ there exists a function $\zeta_r \in W^{1,2}(B_1)$ such that $\zeta_r \geq 0$ in B_1', $\zeta_r = v_r$ on S_1 and

$$W(\zeta_r,1) \leq (1-\kappa)W(w_r,1). \tag{7.7}$$

We note that this inequality continues to hold if as ζ_r we take the minimizer of the functional $W(\cdot,1)$ among all functions $\zeta \in W^{1,2}(B_1)$ with $\zeta\big|_{S_1} = v_r\big|_{S_1}$ and $\zeta \geq 0$ in B_1'. Taking such minimizer is equivalent to saying that ζ_r is the solution of the thin obstacle problem in B_1 for the Laplacian with zero thin obstacle on B_1' and boundary values v_r on S_1. In particular, with this choice of ζ_r

we will have $W(\zeta_r,1)\le W(v_r,1)$. Next,

$$\begin{aligned}W(\zeta_r,1)&=\int_{B_1}|\nabla\zeta_r|^2-\frac{3}{2}\int_{S_1}\zeta_r^2\\&\ge\int_{B_1}\langle A(rx)\nabla\zeta_r,\nabla\zeta_r\rangle-\frac{3}{2}\int_{S_1}\mu(rx)\zeta_r^2-Cr\int_{B_1}|\nabla\zeta_r|^2-Cr\int_{S_1}\zeta_r^2.\end{aligned}$$

If we now let $\hat\zeta=\zeta_r(x/r)r^{3/2}$, then on S_r we have that $\hat\zeta=v_r(x/r)r^{3/2}=v(x)$, and

$$\int_{B_1}\langle A(rx)\nabla\zeta_r,\nabla\zeta_r\rangle-\frac{3}{2}\int_{S_1}\mu(rx)\zeta_r^2=r^{-n-1}\int_{B_r}\langle A\nabla\hat\zeta,\nabla\hat\zeta\rangle-\frac{3}{2}r^{-n-2}\int_{S_r}\mu\hat\zeta^2$$

Since $\hat\zeta=v$ on S_r, $\hat\zeta\ge 0$ on B_r' and v minimizes the energy (1.17) over B_r among all such functions, we obtain

$$\int_{B_r}\left(\langle A\nabla\hat\zeta,\nabla\hat\zeta\rangle+2f\hat\zeta\right)\ge\int_{B_r}\left(\langle A\nabla v,\nabla v\rangle+2fv\right).$$

Next, by (2.16) we have $|v(x)|\le C|x|^{3/2}\le Cr^{3/2}$ in B_r. Besides, noting that $\hat\zeta$ solves the thin obstacle problem in B_r with boundary values v on S_r, by subharmonicity of $\hat\zeta^{\pm}$, we will have that

$$\sup_{B_r}|\hat\zeta|\le\sup_{S_r}v^+ +\sup_{S_r}v^-\le Cr^{3/2}.$$

Hence, we obtain

$$\begin{aligned}\int_{B_r}\langle A\nabla\hat\zeta,\nabla\hat\zeta\rangle&\ge\int_{B_r}\left(\langle A\nabla v,\nabla v\rangle+2fv-2f\hat\zeta\right)\\&\ge\int_{B_r}\langle A\nabla v,\nabla v\rangle-Cr^{n+(3/2)}.\end{aligned}$$

Combining the inequalities above, we have

$$\begin{aligned}W(\zeta,1)&\ge r^{-n-1}\int_{B_r}\langle A\nabla v,\nabla v\rangle-\frac{3}{2}r^{-n-2}\int_{S_r}\mu v^2-Cr\int_{B_1}|\nabla\zeta_r|^2-Cr\int_{S_1}\zeta_r^2-Cr^{1/2}\\&=W_L(v,r)-Cr\left(W(\zeta_r,1)+\frac{3}{2}\int_{S_1}\zeta_r^2\right)-Cr\int_{S_1}\zeta_r^2-Cr^{1/2}\\&\ge W_L(v,r)-Cr\left(W(v_r,1)+\frac{3}{2}\int_{S_1}\zeta_r^2\right)-Cr\int_{S_1}v_r^2-Cr^{1/2}\\&=W_L(v,r)-Cr\left(\int_{B_1}|\nabla v_r|^2-\frac{3}{2}\int_{S_1}v_r^2+\frac{3}{2}\int_{S_1}\zeta_r^2\right)-Cr\int_{S_1}v_r^2-Cr^{1/2}\\&=W_L(v,r)-Cr\int_{B_1}|\nabla v_r|^2-Cr\int_{S_1}v_r^2-Cr^{1/2}\\&\overset{(2.16)}{\ge}W_L(v,r)-Cr^{1/2}.\end{aligned}\tag{7.8}$$

Combining (7.7) and (7.8) we obtain

$$\begin{aligned}W(w_r,1)-W_L(v,r)&\ge\frac{W(\zeta_r,1)}{1-\kappa}-W_L(v,r)\ge\frac{W_L(v,r)-Cr^{1/2}}{1-\kappa}-W_L(v,r)\\&=\frac{\kappa}{1-\kappa}W_L(v,r)-Cr^{1/2}.\end{aligned}\tag{7.9}$$

Therefore, from (7.6) and (7.9) we conclude that

$$
\begin{aligned}
\frac{d}{dr}W_L(v,r) &\geq \frac{n+1}{r}(W(w_r,1) - W_L(v,r)) - Cr^{-1/2} \\
&\geq \frac{n+1}{r}\left(\frac{\kappa}{1-\kappa}W_L(v,r) - Cr^{1/2}\right) - Cr^{-1/2} \\
&\geq \frac{n+1}{r}\frac{\kappa}{1-\kappa}W_L(v,r) - Cr^{-1/2}.
\end{aligned}
\tag{7.10}
$$

Taking $\gamma \in (0, \frac{1}{2} \wedge (n+1)\frac{\kappa}{1-\kappa})$, if we use that $W_L(v,r) \geq -Cr^{1/2}$, see Corollary 4.5, then from the above inequality we will have

$$
\left(W_L(v,r)r^{-\gamma}\right)' \geq -Cr^{-\gamma-1/2}.
$$

Integrating in (t, r_0), for $t \geq s_0$, we obtain

$$
W_L(v,t)t^{-\gamma} \leq W_L(v,r_0)r_0^{-\gamma} + Cr_0^{-\gamma+1/2} - Ct^{-\gamma+1/2}.
$$

This implies, in particular,

$$
W_L(v,t) \leq Ct^{\gamma},
$$

which establishes (7.3).

We now prove (7.4). For a fixed x, define $g(r) = \frac{v(rx)}{r^{3/2}}$, so that $v_t(x) - v_s(x) = g(t) - g(s)$. Then

$$
\begin{aligned}
\int_{S_1} |v_t - v_s| &= \int_{S_1} \left| \int_s^t g'(r)dr \right| \\
&\leq \int_s^t \left(\int_{S_1} r^{-\frac{3}{2}} \left| \langle \nabla v(rx), \nu \rangle - \frac{3}{2}\frac{v(rx)}{r} \right| \right) dr \\
&= \int_s^t \left(r^{-1} \int_{S_1} |\langle \nabla v_r, \nu \rangle - \frac{3}{2}v_r| \right) dr \\
&\leq C_n \int_s^t r^{-1/2} \left(r^{-1} \int_{S_1} (\langle \nabla v_r, \nu \rangle - \frac{3}{2}v_r)^2 \right)^{1/2} dr.
\end{aligned}
\tag{7.11}
$$

By (7.6) and (7.9),

$$
\begin{aligned}
\frac{1}{r}\int_{S_1} (\langle \nabla v_r, \nu \rangle - \frac{3}{2}v_r)^2 &\leq \frac{d}{dr}W_L(v,r) + Cr^{-1/2} - \frac{n+1}{r}(W(w_r,1) - W_L(v,r)) \\
&\leq \frac{d}{dr}W_L(v,r) - \frac{n+1}{r}\frac{k}{1-k}W_L(v,r) + Cr^{-1/2} \\
&\leq \frac{d}{dr}W_L(v,r) + Cr^{-1/2},
\end{aligned}
\tag{7.12}
$$

where we have used again Corollary 4.5, that gives $W_L(v,r) \geq -Cr^{1/2}$. This proves (7.4) and completes the proof. $\square$

The next important step after Lemma 7.1 is contained in Proposition 7.2 below. It proves that the regular set is a relatively open set of the free boundary, and that if $x_0 \in \Gamma_{3/2}(v)$, then for r small enough and $\bar{x} \in \Gamma(v)$ in a small neighborhood of x_0 we can apply Lemma 7.1 to the homogeneous scalings

$$
v_{\bar{x},r}(x) = \frac{v_{\bar{x}}(rx)}{r^{3/2}} = \frac{v(\bar{x} + A^{1/2}(\bar{x})rx) - b_{\bar{x}}rx_n}{r^{3/2}},
$$

which in turn proves the uniqueness of the blowup limits.

Proposition 7.2. *Let v solve (1.12)–(1.15) and $x_0 \in \Gamma_{3/2}(v)$. Then, there exist constants $r_0 = r_0(x_0)$, $\eta_0 = \eta_0(x_0) > 0$ such that $\Gamma(v) \cap B'_{\eta_0}(x_0) \subset \Gamma_{3/2}(v)$. Moreover, if $v_{\bar{x},0}$ is any blowup of v at $\bar{x} \in \Gamma(u) \cap B'_{\eta_0}(x_0)$, as in Definition 5.5, then*

$$\int_{S_1} |v_{\bar{x},r} - v_{\bar{x},0}| \le C r^{\gamma}, \quad \textit{for all } r \in (0, r_0),$$

where C and $\gamma > 0$ are universal constants. In particular, the blowup limit $v_{\bar{x},0}$ is unique.

Proof. Let r_0 and η_0 be as in Lemma 3.4. Then, for $\bar{x} \in B'_{\eta_0}(x_0) \cap \Gamma(v) \subset \Gamma_{3/2}(v)$ and $0 < r < r_0$ consider two scalings, the homogeneous and Almgren types:

$$v_{\bar{x},r}(x) = \frac{v_{\bar{x}}(rx)}{r^{3/2}}, \quad \tilde{v}_{\bar{x},r}(x) = \frac{v_{\bar{x}}(rx)}{d_{\bar{x},r}}$$

By Lemma 3.4, the Almgren scaling $\frac{1}{c_n}\tilde{v}_{\bar{x},r}\big|_{S_1}$ has the epiperimetric property in the sense that if we extend it as a 3/2-homogeneous function in B_1, call it w_r, then Lemma 3.4 allows us to apply the epiperimetric inequality to conclude that there exists $\zeta_r \in W^{1,2}(B_1)$ such that $\zeta_r = \frac{1}{c_n}\tilde{v}_{\bar{x},r}$ on S_1, $\zeta_r \ge 0$ in B'_1 and

$$W(\zeta_r) \le (1-\kappa) W(w_r).$$

We next observe that if a certain function on S_1 has the epiperimetric property then so does any of its multiples. In particular, $v_{\bar{x},r}\big|_{S_1} = \frac{c_n d_{\bar{x},r}}{r^{3/2}} \frac{1}{c_n}\tilde{v}_{\bar{x},r}$ has the epiperimetric property for any $\bar{x} \in B'_{\eta_0}(x_0) \cap \Gamma(v)$ and $r < r_0$. Thus, we can apply Lemma 7.1 to conclude that

$$\int_{S_1} |v_{\bar{x},r} - v_{\bar{x},s}| \le C r^{\gamma}, \quad \text{for } 0 < s \le r < r_0,$$

for universal C and $\gamma > 0$. Now, if over some sequence $v_{\bar{x},s_j} \to v_{\bar{x},0}$ (see Lemma 5.2), we will obtain that

$$\int_{S_1} |v_{\bar{x},r} - v_{\bar{x},0}| \le C r^{\gamma}, \quad \text{for } 0 < r < r_0.$$

□

We notice explicitly that up to this point we have not excluded the possibility that the blowup $v_{\bar{x},0} \equiv 0$. This is done in the following proposition.

Proposition 7.3. *The unique blowup $v_{\bar{x},0}$ in Proposition 7.2 is nonzero.*

Proof. Assume to the contrary that $v_{\bar{x},0} = 0$. Then, from Proposition 7.2 we have that

$$\int_{S_1} |v_{\bar{x},r}| \le C r^{\gamma}, \quad \text{for } 0 < r < r_0.$$

But then,

$$\int_{S_1} |\tilde{v}_{\bar{x},r}| = \int_{S_1} |v_{\bar{x},r}| \frac{r^{3/2}}{d_{\bar{x},r}} \le C \frac{r^{3/2+\gamma}}{d_{\bar{x},r}}. \tag{7.13}$$

On the other hand, by Lemma 3.2 for every $\varepsilon > 0$ there exists $r_\varepsilon > 0$ such that

$$d_{\bar{x},r} = \left(\frac{1}{r^{n-1}} H_{\bar{x}}(r)\right) \ge c M_{\bar{x}}(r)^{1/2} \ge c r^{(3+\varepsilon)/2}, \quad \text{for } 0 < r < r_\varepsilon$$

If we choose $\varepsilon < 2\gamma$, we then obtain as $r \to 0+$

$$\int_{S_1} |\tilde{v}_{\bar{x},r}| \le C r^{\gamma - \varepsilon/2} \to 0.$$

Since there exists $e' \in \mathbb{R}^{n-1}$, and a subsequence $r = r_j \to 0+$ such that $\tilde{v}_{\bar{x},r} \to c_n \Re(\langle x', e'\rangle + i|x_n|)^{3/2}$, this is clearly a contradiction. □

In what follows we denote by

$$v_{\bar{x},0}(x) = a_{\bar{x}} \Re(\langle x', \nu_{\bar{x}}\rangle + i|x_n|^{3/2}), \quad a_{\bar{x}} > 0,\ \nu_{\bar{x}} \in S_1'$$

the unique homogeneous blowup at $\bar{x}$.

Proposition 7.4. *Let v be a solution of* (1.12)–(1.15) *with $x_0 \in \Gamma_{3/2}(v)$. Then, there exists $\eta_0 > 0$ depending on x_0 such that*

$$\int_{S_1'} |v_{\bar{x},0} - v_{\bar{y},0}| \le C|\bar{x} - \bar{y}|^\beta \quad \text{for } \bar{x}, \bar{y} \in B'_{\eta_0}(x_0) \cap \Gamma(v),$$

where C and $\beta > 0$ are universal constants.

Proof. Let η_0 and r_0 be as in Proposition 7.2. Then, we will have

$$\int_{S_1} |v_{\bar{x},0} - v_{\bar{y},0}| \le Cr^\gamma + \int_{S_1} |v_{\bar{x},r} - v_{\bar{y},r}|$$

for any $r < r_0$ and $\bar{x}, \bar{y} \in B_{\eta_0} \cap \Gamma(v)$. In this inequality we will chose $r = |\bar{x} - \bar{y}|^\sigma$ with $0 < \sigma < 1$ to be specified below. We then estimate the integral on the right-hand side of the above inequality. First we notice that

$$\begin{aligned} v_{\bar{x}}(z) &= v(\bar{x} + A^{1/2}(\bar{x})z) - \langle A^{1/2}(\bar{x})\nabla v(\bar{x}), e_n\rangle z_n \\ &= v(\bar{x} + A^{1/2}(\bar{x})z) - \partial_n v(\bar{x})\langle \bar{x} + A^{1/2}(\bar{x})z, e_n\rangle \end{aligned}$$

by using property (1.5) of the coefficient matrix A. Therefore, denoting

$$\begin{aligned} \xi(s) &= [s\bar{x} + (1-s)\bar{y}] + [sA^{1/2}(\bar{x}) + (1-s)A^{1/2}(\bar{y})]z \\ p(s) &= s\partial_n v(\bar{x}) + (1-s)\partial_n v(\bar{y}) \end{aligned}$$

we will have

$$\begin{aligned} v_{\bar{x}}(z) - v_{\bar{y}}(z) &= \int_0^1 \frac{d}{ds}\left(v(\xi(s)) - p(s)\langle \xi(s), e_n\rangle\right) ds \\ &= \int_0^1 \big(\langle \nabla v(\xi(s)) - p(s)e_n, [\bar{x} - \bar{y}] + [A^{1/2}(\bar{x}) - A^{1/2}(\bar{y})]z\rangle \\ &\qquad - [\partial_n v(\bar{x}) - \partial_n v(\bar{y})]\langle [sA^{1/2}(\bar{x}) + (1-s)A^{1/2}(\bar{y})]z, e_n\rangle\big) ds \end{aligned}$$

where for the last term we have used the orthogonality $\langle [s\bar{x} + (1-s)\bar{y}], e_n\rangle = 0$. This gives

$$|v_{\bar{x}}(z) - v_{\bar{y}}(z)| \le C(|\bar{x} - \bar{y}| + |z|)^{1/2}(|\bar{x} - \bar{y}| + |z||\bar{x} - \bar{y}|) + C|\bar{x} - \bar{y}|^{1/2}|z|.$$

Using the above estimate we then obtain

$$\begin{aligned} \int_{S_1} |v_{\bar{x},r} - v_{\bar{y},r}| &= \int_{S_1} \frac{|v_{\bar{x}}(rz) - v_{\bar{y}}(rz)|}{r^{3/2}} \\ &\le C\frac{(|\bar{x} - \bar{y}| + r)^{1/2}|\bar{x} - \bar{y}|(1+r) + |\bar{x} - \bar{y}|^{1/2} r}{r^{3/2}} \\ &\le C|\bar{x} - \bar{y}|^{1-\sigma} + C|\bar{x} - \bar{y}|^{(1-\sigma)/2} \le C|\bar{x} - \bar{y}|^{(1-\sigma)/2}, \end{aligned}$$

if we choose $r = |\bar{x} - \bar{y}|^\sigma$ with $0 < \sigma < 1$.

Going back to the beginning of the proof, we conclude

$$\int_{S_1} |v_{\bar{x},0} - v_{\bar{y},0}| \le C(|\bar{x} - \bar{y}|^{\sigma\gamma} + |\bar{x} - \bar{y}|^{(1-\sigma)/2}) = C|\bar{x} - \bar{y}|^{2\beta},$$

with $2\beta = \gamma/(1+2\gamma)$ if we choose $\sigma = 1/(1+2\gamma)$.

It remains to show that we can change the integration over $(n-1)$-dimensional sphere S_1 to $(n-2)$-dimensional S_1'. To this end, we note that both $v_{\bar{x},0}$ and $v_{\bar{y},0}$ are solutions of the Signorini problem for the Laplacian with zero thin obstacle and therefore

$$\Delta(v_{\bar{x},0} - v_{\bar{y},0})_{\pm} \ge 0^{*}.$$

Thus by the energy inequality we obtain that

$$\int_{B_1} |\nabla(v_{\bar{x},0} - \nabla v_{\bar{y},0})|^2 \le C \int_{B_1} |(v_{\bar{x},0} - v_{\bar{y},0})|^2 \le C|\bar{x} - \bar{y}|^{2\beta}.$$

(Recall that $v_{\bar{z},0}$ are 3/2-homogeneous functions with uniform $C^{1,1/2}$ estimates). Then, using the trace inequality, we obtain that

$$\int_{B'_{1/2}} |v_{\bar{x},0} - v_{\bar{y},0}|^2 \le C|\bar{x} - \bar{y}|^{2\beta}$$

This is equivalent to

$$\int_{S'_1} |v_{\bar{x},0} - v_{\bar{y},0}|^2 \le C|\bar{x} - \bar{y}|^{2\beta},$$

and using Hölder's inequality

$$\int_{S'_1} |v_{\bar{x},0} - v_{\bar{y},0}| \le C|\bar{x} - \bar{y}|^{\beta},$$

as claimed. □

Lemma 7.5. *Let v be as in Proposition 7.4 and $v_{\bar{x},0} = a_{\bar{x}}\Re(\langle x', \nu_{\bar{x}}\rangle + i|x_n|)^{3/2}$ be the unique homogeneous blowup at $\bar{x} \in B'_{\eta_0}(x_0) \cap \Gamma(v)$. Then, for a constant C_0 depending on x_0 we have*

$$|a_{\bar{x}} - a_{\bar{y}}| \le C_0|\bar{x} - \bar{y}|^{\beta} \tag{7.14}$$

$$|\nu_{\bar{x}} - \nu_{\bar{y}}| \le C_0|\bar{x} - \bar{y}|^{\beta}, \tag{7.15}$$

for $\bar{x}, \bar{y} \in B'_{\eta_0}(x_0) \cap \Gamma(v)$.

Proof. The first inequality follows from the observation that

$$C_n a_{\bar{x}} = \|v_{\bar{x},0}\|_{L^1(S'_1)}, \quad C_n a_{\bar{y}} = \|v_{\bar{y},0}\|_{L^1(S'_1)}$$

with the same dimensional constant C_n and therefore

$$C_n|a_{\bar{x}} - a_{\bar{y}}| \le \|v_{\bar{x},0} - v_{\bar{y},0}\|_{L^1(S'_1)} \le C|\bar{x} - \bar{y}|^{\beta},$$

which establishes (7.14). To prove (7.15), we first note that by (7.14)

$$\|a_{\bar{x}}\langle z, \nu_{\bar{x}}\rangle_+^{3/2} - a_{\bar{y}}\langle z, \nu_{\bar{y}}\rangle_+^{3/2}\|_{L^1(S'_1)} \le C|\bar{x} - \bar{y}|^{\beta}$$

and therefore we obtain

$$\|\langle z, \nu_{\bar{x}}\rangle_+^{3/2} - \langle z, \nu_{\bar{y}}\rangle_+^{3/2}\|_{L^1(S'_1)} \le C_0|\bar{x} - \bar{y}|^{\beta}.$$

(Here we have used that by (7.14) we may assume that $a_{\bar{y}} > a_{x_0}/2$ if η_0 is small).

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* *Short proof:* the only nontrivial place to verify the subharmonicity of $(v_{\bar{x},0} - v_{\bar{y},0})_+$ is at points $z \in \mathbb{R}^{n-1}$ with $v_{\bar{x},0}(z) > 0$ and $v_{\bar{y},0}(z) = 0$; but near such z, $\Delta v_{\bar{x},0} = 0$ and $\Delta v_{\bar{y},0} \le 0$ by the Signorini conditions on $\mathbb{R}^{n-1}$.

On the other hand, it is easy to see from geometric considerations[†] that

$$\|\langle z,\nu_{\bar x}\rangle_+^{3/2}-\langle z,\nu_{\bar y}\rangle_+^{3/2}\|_{L^1(S_1')}\geq c_n|\nu_{\bar x}-\nu_{\bar y}|$$

implying that

$$c_n|\nu_{\bar x}-\nu_{\bar y}|\leq C_0|\bar x-\bar y|^\beta.$$

□

Theorem 7.6. *Let v be a solution of (1.12)–(1.15) with $x_0\in\Gamma_{3/2}(v)$. Then there exists a positive η_0 depending on x_0 such that $B'_{\eta_0}(x_0)\cap\Gamma(v)\subset\Gamma_{3/2}(v)$ and*

$$B'_{\eta_0}\cap\Lambda(v)=B'_{\eta_0}\cap\{x_{n-1}\leq g(x'')\}$$

for $g\in C^{1,\beta}(\mathbb{R}^{n-2})$ with a universal exponent $\beta\in(0,1)$, after a possible rotation of coordinate axes in $\mathbb{R}^{n-1}$.

Proof. We subdivide the proof into several steps.

Step 1. Let η_0 be as Proposition 7.2. We then claim that for any $\varepsilon>0$ there is $r_\varepsilon>0$ such that

$$\|v_{\bar x,r}-v_{\bar x,0}\|_{C^1(B_1^\pm\cup B_1')}<\varepsilon,\quad\text{for } \bar x\in B'_{\eta_0/2}(x_0)\cap\Gamma(v),\ r<r_\varepsilon.$$

Indeed, arguing by contradiction, we will have a sequence of points $\bar x_j\in B'_{\eta_0/2}(x_0)\cap\Gamma(v)$ and radii $r_j\to0$ such that

$$\|v_{\bar x_j,r_j}-v_{\bar x_j,0}\|_{C^1(B_1^\pm\cup B_1')}\geq\varepsilon_0$$

for some $\varepsilon_0>0$. Clearly, we may assume that $\bar x_j\to\bar x_0\in\overline{B'_{\eta_0/2}(x_0)}\cap\Gamma(v)$. Now, the scalings $v_{\bar x_j,r_j}$ are uniformly bounded in $C^{1,1/2}(B_R^\pm\cup B_R')$ for any $R>0$ and thus we may assume that

$$v_{\bar x_j,r_j}\to w\quad\text{in } C^1_{\mathrm{loc}}((\mathbb{R}^n)^\pm\cup\mathbb{R}^{n-1}\}.$$

We claim that actually $w=v_{\bar x_0,0}$. Indeed, by integrating the inequality in Proposition 7.2, we will have

$$\|v_{\bar x,r}-v_{\bar x,0}\|_{L^1(B_R)}\leq C_Rr^\gamma,\quad\text{for } \bar x\in B'_{\eta_0}(x_0)\cap\Gamma(v),\ r<r_0/R,$$

which will immediately imply that $v_{\bar x_j,0}\to w$ in $L^1(B_R)$. On the other hand, Lemma 7.5 implies that $v_{\bar x_j,0}\to v_{\bar x_0,0}$ in $C^1(B_R^\pm\cup B_R')$. Hence $w=v_{\bar x_0,0}$. Moreover, we get that both $v_{\bar x_j,r_j}$ and $v_{\bar x_j,0}$ converge in $C^1(B_1^\pm\cup B_1')$ to the same function $v_{\bar x_0,0}$ and therefore

$$\|v_{\bar x_j,r_j}-v_{\bar x_j,0}\|_{C^1(B_1^\pm\cup B_1')}\to0$$

contrary to our assumption.

Step 2. For a given $\varepsilon>0$ and a unit vector $\nu\in\mathbb{R}^{n-1}$ define the cone

$$\mathcal{C}_\varepsilon(\nu)=\{x'\in\mathbb{R}^{n-1}\mid\langle x',\nu\rangle\geq\varepsilon|x'|\}$$

We then claim that for any $\varepsilon>0$ there exists r_ε such that for any $\bar x\in B'_{\eta_0/2}(x_0)\cap\Gamma(v)$ we have

$$\mathcal{C}_\varepsilon(\nu_{\bar x})\cap B'_{r_\varepsilon}\subset\{v_{\bar x}(\cdot,0)>0\}.$$

Indeed, consider a cutout from the sphere $S'_{1/2}$ by the cone $\mathcal{C}_\varepsilon(\nu)$

$$K_\varepsilon(\nu)=\mathcal{C}_\varepsilon(\nu)\cap S'_{1/2}.$$

Note that

$$K_\varepsilon(\nu_{\bar x})\Subset\{v_{\bar x,0}(\cdot,0)>0\}\cap B_1'\quad\text{and}\quad v_{\bar x,0}(\cdot,0)\geq a_{\bar x}c_\varepsilon\quad\text{on } K_\varepsilon(\nu_{\bar x})$$

[†]just notice that $\partial_\theta\int_{\substack{\langle z,e_1\rangle\geq1/2\\ \langle z,e_2\rangle\geq1/2}}\langle z,\cos\theta e_1+\sin\theta e_2\rangle^{3/2}\Big|_{\theta=0}=\frac32\int_{\substack{\langle z,e_1\rangle\geq1/2\\ \langle z,e_2\rangle\geq1/2}}\langle z,e_1\rangle^{1/2}\langle z,e_2\rangle>0$

for some universal $c_\varepsilon > 0$. Without loss of generality, by Lemma 7.5, we may assume that $a_{\bar{x}} \geq a_0/2$ for $\bar{x} \in B'_{\eta_0}(x_0) \cap \Gamma(v)$. Then, applying Step 1 above, we can find $r_\varepsilon > 0$ such that

$$v_{\bar{x},r}(\cdot,0) > 0 \quad \text{on } K_\varepsilon(\nu_{\bar{x}}), \quad \text{for all } r < r_\varepsilon$$

Scaling back by r, we have

$$v_{\bar{x}}(\cdot,0) > 0 \quad \text{on} \quad rK_\varepsilon(\nu_{\bar{x}}) = \mathcal{C}_\varepsilon(\nu) \cap S'_{r/2}, \quad r < r_\varepsilon$$

Taking the union over all $r < r_\varepsilon$, we obtain

$$\mathcal{C}_\varepsilon(\nu) \cap B'_{r_\varepsilon/2} \subset \{v_{\bar{x}}(\cdot,0) > 0\}.$$

Step 3. We next claim that for any $\varepsilon > 0$ there exists r_ε such that for any $\bar{x} \in B'_{\eta_0/2}(x_0) \cap \Gamma(v)$ we have

$$-\mathcal{C}_\varepsilon(\nu_{\bar{x}}) \cap B'_{r_\varepsilon} \subset \{v_{\bar{x}}(\cdot,0) = 0\}.$$

To prove that we note that

$$-K_\varepsilon(\nu_{\bar{x}}) \Subset \{v_{\bar{x},0}(\cdot,0) = 0\} \cap B'_1 \quad \text{and} \quad (\partial^-_{x_n} - \partial^+_{x_n}) v_{\bar{x},0}(\cdot,0) \geq a_{\bar{x}} c_\varepsilon > (a_0/2)c_\varepsilon \quad \text{on } -K_\varepsilon(\nu_{\bar{x}})$$

for a universal $c_\varepsilon > 0$. Then, from Step 1, we will have the existence of $r_\varepsilon > 0$ such that

$$\langle A_{\bar{x}}(rx)\nabla v_{\bar{x},r}(x), e_n^-\rangle + \langle A_{\bar{x}}(rx)\nabla v_{\bar{x},r}(x), e_n^+\rangle > 0 \quad \text{on } -K_\varepsilon(\nu_{\bar{x}}), \quad \text{for all } r < r_\varepsilon.$$

Thus, from the complementarity conditions, we will have

$$v_{\bar{x},r}(\cdot,0) = 0 \quad \text{on } -K_\varepsilon(\nu_{\bar{x}}), \quad \text{for all } r < r_\varepsilon.$$

Arguing as in the end of Step 2, we conclude that

$$-\mathcal{C}_\varepsilon(\nu) \cap B'_{r_\varepsilon/2} \subset \{v_{\bar{x}}(\cdot,0) = 0\}.$$

Step 4. Here without loss of generality we will assume that $A(x_0) = I$ and $\nu_{x_0} = e_{n-1}$. Changing from function $v_{\bar{x}}$ to v, we may rewrite the result of Steps 2 and 3 as

$$\begin{aligned} \bar{x} + A(\bar{x})^{1/2}(\mathcal{C}_\varepsilon(\nu_{\bar{x}}) \cap B'_{r_\varepsilon/2}) &\subset \{v(\cdot,0) > 0\}, \\ \bar{x} - A(\bar{x})^{1/2}(\mathcal{C}_\varepsilon(\nu_{\bar{x}}) \cap B'_{r_\varepsilon/2}) &\subset \{v(\cdot,0) = 0\}, \end{aligned}$$

for any $\bar{x} \in B'_{\eta_0}(x_0) \cap \Gamma(v)$. Taking $\bar{x}$ sufficiently close to x_0 (and using Lemma 7.5) we can guarantee that

$$A(\bar{x})^{1/2}(\mathcal{C}_\varepsilon(\nu_{\bar{x}}) \cap B'_{r_\varepsilon/2}) \supset \mathcal{C}_{2\varepsilon}(e_{n-1}) \cap B'_{r_\varepsilon/4}.$$

Hence, there exists $\eta_\varepsilon > 0$ such that

$$\begin{aligned} \bar{x} + (\mathcal{C}_{2\varepsilon}(e_{n-1}) \cap B'_{r_\varepsilon/4}) &\subset \{v(\cdot,0) > 0\}, \\ \bar{x} - (\mathcal{C}_{2\varepsilon}(e_{n-1}) \cap B'_{r_\varepsilon/4}) &\subset \{v(\cdot,0) = 0\} \end{aligned} \tag{7.16}$$

for any $\bar{x} \in B'_{\eta_\varepsilon}(x_0) \cap \Gamma(v)$. Now, fixing $\varepsilon = \varepsilon_0$, by the standard arguments, we can conclude that there exists a Lipschitz function $g : \mathbb{R}^{n-2} \to \mathbb{R}$ with $|\nabla g| \leq C_n/\varepsilon_0$ such that

$$\begin{aligned} B'_{\eta_{\varepsilon_0}}(x_0) \cap \{v(\cdot,0) = 0\} &= B'_{\eta_{\varepsilon_0}}(x_0) \cap \{x_{n-1} \leq g(x'')\} \\ B'_{\eta_{\varepsilon_0}}(x_0) \cap \{v(\cdot,0) > 0\} &= B'_{\eta_{\varepsilon_0}}(x_0) \cap \{x_{n-1} > g(x'')\} \end{aligned}$$

Step 5. Using the normalization $A(x_0) = I$ and $\nu_{x_0} = e_{n-1}$ as in Step 4 and letting $\varepsilon \to 0$ we see that $\Gamma(v)$ is differentiable at x_0 with normal ν_0. Recentering at any $\bar{x} \in B'_{\eta_{\varepsilon_0}}(x_0) \cap \Gamma(v)$, we see that $\Gamma(v)$ has a normal

$$A(\bar{x})^{-1/2}\nu_{\bar{x}}$$

at $\bar{x}$. Finally noting that by Lemma 7.5 the mapping $\bar{x} \mapsto A(\bar{x})^{-1/2}\nu_{\bar{x}}$ is C^{β}, we obtain that the function g in Step 4 is $C^{1,\beta}$.

The proof is complete. □

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